Schellbach-style Formulae for the Derousseau-Pampuch Generalizations of the Malfatti Circles

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Abstract

It is known that there exist 32 triplets of circles such that each circle is tangent to the other two circles and to two of the sides of the triangle or their extensions. We provide formulae to obtain the radii of the circles for each of the 32 triplets from the side lengths of the reference triangle by means of trigonometric or hyperbolic functions.

1 Introduction

The configuration of three circles inside a triangle such that each circle is tangent to the other two circles and to two of the sides of the triangle has been studied for more than two centuries. Today, such three circles are called the *Malfatti circles* of the triangle.

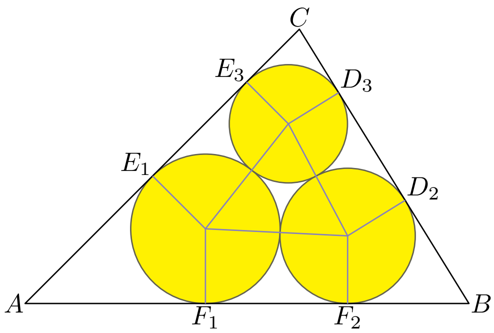

Figure 1:

Sometime before 1773, Naonobu Ajima (1732?–1798), who was a samurai, or a member of the military class in old Japan, found a method to calculate the diameters of the Malfatti circles from the side lengths of an arbitrary triangle. The method was called *Nanzan-shi san-sha naiyō san-en jutsu* (“Nanzan's

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method on a triangle that includes three circles", as Nanzan is a pen name of Ajima's) or *San-sha san-en jutsu* in short. A brief description of the method is found in [2, I ¶14]. A detailed description of the method including a proof is found in [1]. Unfortunately, Ajima's method as well as any other results by Japanese mathematicians in those days was inaccessible from outside Japan until the Edo shogunate, the former government of Japan (1603–1868), abandoned the isolation policy in 1854.

In 1803, an Italian mathematician Gianfrancesco Malfatti (1731–1807) [10] gave a construction to draw the Malfatti circles for an arbitrary triangle. Despite Malfatti's unawareness of Ajima's works, Malfatti's construction is considered identical in many parts to Ajima's method.

In 1852, Schellbach [12][13] gave a set of formulae to obtain the distances between the vertices and the tangent points of the circles on the sides from the side lengths of an arbitrary triangle by using trigonometric functions. The same formulae with a proof essentially identical to Schellbach's are described in English in [6, §30][7].

In 1895, Derousseau [4] generalized the Malfatti circles by removing the condition that the three circles are inside the triangle. Derousseau proved that there exist 32 triplets of circles such that each circle is tangent to the other two circles and to two of the sides of the reference triangle or their extensions. Some alternative proofs of the existence are known. In 1904, Pampuch [11] gave another proof. In 1930, Lob and Richmond [9] gave yet another proof.

In this article, we provide formulae to obtain the radii of the circles for each of the 32 triplets from the side lengths by means of trigonometric or hyperbolic functions. In other words, we provide Schellbach-style formulae for all of the Derousseau-Pampuch generalizations.

2 Notation

Throughout this article, we use the following notation.

For a triangle ABC, let a, b, c denote the lengths of the sides BC, CA, AB, s the semiperimeter, r the inradius, and r_A, r_B, r_C the exradii as usual.

Let the incircle is tangent to the side BC at D, to the side CA at E, and to the side AB at F. Let the excircle corresponding to the vertex A is tangent to the side BC at D_A, to the extension of the side AC at E_A, and to the extension of the side AB at F_A.

Suppose the circle $A'(r_1)$ is tangent to the line CA at E_1 and to the line AB at F_1, the circle $B'(r_2)$ is tangent to the line AB at F_2 and to the line BC at D_2, the circle $C'(r_3)$ is tangent to the line BC at D_3 and to the line CA at E_3, and the three circles are tangent to one another. Suppose the nine tangent points are distinct.

3 Classification

Since the center A' does not locate neither on the line AB nor on the line AC, it locates inside $\angle CAB$, inside $\angle \overline{C}AB$, inside $\angle CA\overline{B}$ or inside $\angle \overline{C}A\overline{B}$ where an overline indicates that the angle has, as one of its sides, the opposite ray instead of the ray including the triangle side. For example, $\angle CA\overline{B}$ denotes the

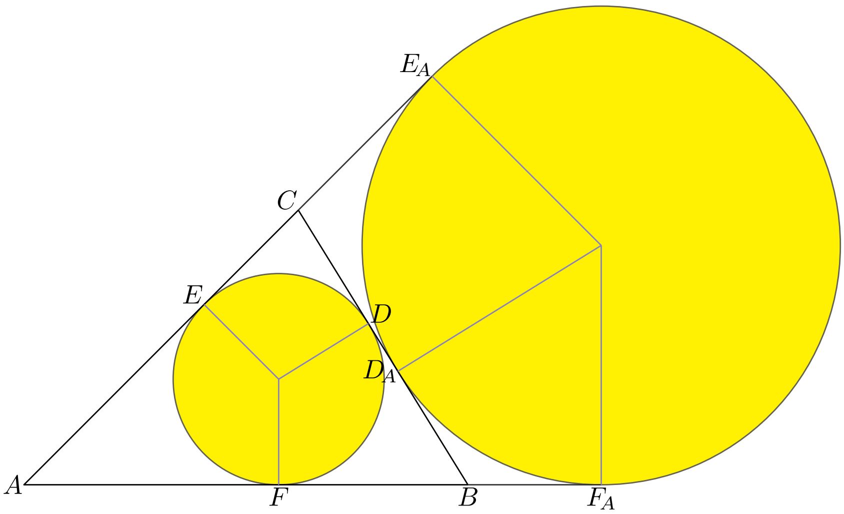

Figure 2:

angle with the ray $\overrightarrow{AC}$ and the ray opposite to the ray $\overrightarrow{B}$. And $\angle\overline{C}A\overline{B}$ denotes the vertical angle of $\angle CAB$. Analogously, the center B' locates inside $\angle ABC$, inside $\angle\overline{A}BC$, inside $\angle AB\overline{C}$ or inside $\angle\overline{A}B\overline{C}$ and the center C' locates inside $\angle BCA$, inside $\angle\overline{B}CA$, inside $\angle BC\overline{A}$ or inside $\angle\overline{B}C\overline{A}$.

If the circles $A'(r_1)$, $B'(r_2)$, $C'(r_3)$ lie in Δ_1, Δ_2, Δ_3, respectively, then $\Delta_1 \cap \Delta_2 \neq \emptyset$, $\Delta_1 \cap \Delta_3 \neq \emptyset$, and $\Delta_2 \cap \Delta_3 \neq \emptyset$ since the three circles are tangent to one another. Thus, for locations of the three centers A', B', C', only 7 out of the 64 cases are consistent to the condition that the circles are tangent to one another. The following are the consistent cases.

	A' is inside	B' is inside	C' is inside
Case 1	$\angle CAB$	$\angle ABC$	$\angle BCA$
Case 2	$\angle CAB$	$\angle\overline{A}BC$	$\angle BC\overline{A}$
Case 3	$\angle\overline{C}A\overline{B}$	$\angle AB\overline{C}$	$\angle\overline{B}CA$
Case 4	$\angle CA\overline{B}$	$\angle ABC$	$\angle\overline{B}CA$
Case 5	$\angle\overline{C}AB$	$\angle\overline{A}B\overline{C}$	$\angle BC\overline{A}$
Case 6	$\angle\overline{C}AB$	$\angle AB\overline{C}$	$\angle BCA$
Case 7	$\angle CA\overline{B}$	$\angle\overline{A}BC$	$\angle\overline{B}C\overline{A}$

4 Solution

4.1 Case 1

In Case 1, the following three conditions hold.

$$
\begin{aligned}
BD_2 + D_3C + D_2D_3 = BD + DC \quad &\text{or} \quad BD_2 + D_3C - D_2D_3 = BD + DC,\\
AE_1 + E_3C + E_1E_3 = AE + EC \quad &\text{or} \quad AE_1 + E_3C - E_1E_3 = AE + EC,\\
AF_1 + F_2B + F_1F_2 = AF + FB \quad &\text{or} \quad AF_1 + F_2B - F_1F_2 = AF + FB.
\end{aligned}
$$

By expressing the lengths by the radii and the angle sizes, we obtain from the first disjunction that

$$r_2 \cot \frac{B}{2} + r_3 \cot \frac{C}{2} + 2\sqrt{r_2 r_3} = r \cot \frac{B}{2} + r \cot \frac{C}{2} \tag{1}$$

or

$$r_2 \cot \frac{B}{2} + r_3 \cot \frac{C}{2} - 2\sqrt{r_2 r_3} = r \cot \frac{B}{2} + r \cot \frac{C}{2}, \tag{2}$$

we obtain from the second disjunction that

$$r_1 \cot \frac{A}{2} + r_3 \cot \frac{C}{2} + 2\sqrt{r_1 r_3} = r \cot \frac{A}{2} + r \cot \frac{C}{2} \tag{3}$$

or

$$r_1 \cot \frac{A}{2} + r_3 \cot \frac{C}{2} - 2\sqrt{r_1 r_3} = r \cot \frac{A}{2} + r \cot \frac{C}{2}, \tag{4}$$

and we obtain from the second disjunction that

$$r_1 \cot \frac{A}{2} + r_2 \cot \frac{B}{2} + 2\sqrt{r_1 r_2} = r \cot \frac{A}{2} + r \cot \frac{B}{2} \tag{5}$$

or

$$r_1 \cot \frac{A}{2} + r_2 \cot \frac{B}{2} - 2\sqrt{r_1 r_2} = r \cot \frac{A}{2} + r \cot \frac{B}{2}. \tag{6}$$

Define l, m, n by

$$l = \cot \frac{A}{2}, \qquad m = \cot \frac{B}{2}, \qquad n = \cot \frac{C}{2}. \tag{7}$$

Define u, v, w, x, y, z by

$$u = \begin{cases} \dfrac{\sqrt{r_2 r_3}}{r} & \text{if (1) holds,} \\ -\dfrac{\sqrt{r_2 r_3}}{r} & \text{if (2) holds,} \end{cases}$$
$$v = \begin{cases} \dfrac{\sqrt{r_1 r_3}}{r} & \text{if (3) holds,} \\ -\dfrac{\sqrt{r_1 r_3}}{r} & \text{if (4) holds,} \end{cases}$$
$$w = \begin{cases} \dfrac{\sqrt{r_1 r_2}}{r} & \text{if (5) holds,} \\ -\dfrac{\sqrt{r_1 r_2}}{r} & \text{if (6) holds,} \end{cases}$$
$$x = \frac{r_1}{r}, \qquad y = \frac{r_2}{r}, \qquad z = \frac{r_3}{r}.$$

Then we have

$$\begin{cases} my + nz + 2u = m + n, \\ lx + nz + 2v = l + n, \\ lx + my + 2w = l + m, \\ xy = w^2, \\ xz = v^2, \\ yz = u^2. \end{cases} \tag{8}$$

For any triangle ABC, if l, m, n are defined by (7), then $lmn = l + m + n$ holds. On the other hand, if positive reals l, m, n satisfy $lmn = l + m + n$, then there exists a triangle ABC that satisfies (7). Thus, Case 1 can be reduced into solving the system of equations (8) for u, v, w, x, y, z with positive real parameters l, m, n under the restriction $lmn = l + m + n$.

As we will show in Appendix A, the system of equations has the following 8 solutions.

$$\begin{cases} u = \dfrac{\sqrt{l^2+1} - l + 1}{2}, \\ v = \dfrac{\sqrt{m^2+1} - m + 1}{2}, \\ w = \dfrac{\sqrt{n^2+1} - n + 1}{2}, \\ x = \dfrac{l + m + n - 1 + \sqrt{l^2+1} - \sqrt{m^2+1} - \sqrt{n^2+1}}{2l}, \\ y = \dfrac{l + m + n - 1 - \sqrt{l^2+1} + \sqrt{m^2+1} - \sqrt{n^2+1}}{2m}, \\ z = \dfrac{l + m + n - 1 - \sqrt{l^2+1} - \sqrt{m^2+1} + \sqrt{n^2+1}}{2n}. \end{cases} \tag{9}$$

$$\begin{cases} u = \dfrac{\sqrt{l^2+1} - l + 1}{2}, \\ v = -\dfrac{\sqrt{m^2+1} + m - 1}{2}, \\ w = -\dfrac{\sqrt{n^2+1} + n - 1}{2}, \\ x = \dfrac{l + m + n - 1 + \sqrt{l^2+1} + \sqrt{m^2+1} + \sqrt{n^2+1}}{2l}, \\ y = \dfrac{l + m + n - 1 - \sqrt{l^2+1} - \sqrt{m^2+1} + \sqrt{n^2+1}}{2m}, \\ z = \dfrac{l + m + n - 1 - \sqrt{l^2+1} + \sqrt{m^2+1} - \sqrt{n^2+1}}{2n}. \end{cases} \tag{10}$$

$$\begin{cases} u = -\dfrac{\sqrt{l^2+1} + l - 1}{2}, \\ v = \dfrac{\sqrt{m^2+1} - m + 1}{2}, \\ w = -\dfrac{\sqrt{n^2+1} + n - 1}{2}, \\ x = \dfrac{l + m + n - 1 - \sqrt{l^2+1} - \sqrt{m^2+1} + \sqrt{n^2+1}}{2l}, \\ y = \dfrac{l + m + n - 1 + \sqrt{l^2+1} + \sqrt{m^2+1} + \sqrt{n^2+1}}{2m}, \\ z = \dfrac{l + m + n - 1 + \sqrt{l^2+1} - \sqrt{m^2+1} - \sqrt{n^2+1}}{2n}. \end{cases} \tag{11}$$

$$
\begin{cases}
u = -\dfrac{\sqrt{l^2+1}+l-1}{2}, \\
v = -\dfrac{\sqrt{m^2+1}+m-1}{2}, \\
w = \dfrac{\sqrt{n^2+1}-n+1}{2}, \\
x = \dfrac{l+m+n-1-\sqrt{l^2+1}+\sqrt{m^2+1}-\sqrt{n^2+1}}{2l}, \\
y = \dfrac{l+m+n-1+\sqrt{l^2+1}-\sqrt{m^2+1}-\sqrt{n^2+1}}{2m}, \\
z = \dfrac{l+m+n-1+\sqrt{l^2+1}+\sqrt{m^2+1}+\sqrt{n^2+1}}{2n}.
\end{cases} \tag{12}
$$

$$
\begin{cases}
u = -\dfrac{\sqrt{l^2+1}+l+1}{2}, \\
v = -\dfrac{\sqrt{m^2+1}+m+1}{2}, \\
w = -\dfrac{\sqrt{n^2+1}+n+1}{2}, \\
x = \dfrac{l+m+n+1-\sqrt{l^2+1}+\sqrt{m^2+1}+\sqrt{n^2+1}}{2l}, \\
y = \dfrac{l+m+n+1+\sqrt{l^2+1}-\sqrt{m^2+1}+\sqrt{n^2+1}}{2m}, \\
z = \dfrac{l+m+n+1+\sqrt{l^2+1}+\sqrt{m^2+1}-\sqrt{n^2+1}}{2n}.
\end{cases} \tag{13}
$$

$$
\begin{cases}
u = -\dfrac{\sqrt{l^2+1}+l+1}{2}, \\
v = \dfrac{\sqrt{m^2+1}-m-1}{2}, \\
w = \dfrac{\sqrt{n^2+1}-n-1}{2}, \\
x = \dfrac{l+m+n+1-\sqrt{l^2+1}-\sqrt{m^2+1}-\sqrt{n^2+1}}{2l}, \\
y = \dfrac{l+m+n+1+\sqrt{l^2+1}+\sqrt{m^2+1}-\sqrt{n^2+1}}{2m}, \\
z = \dfrac{l+m+n+1+\sqrt{l^2+1}-\sqrt{m^2+1}+\sqrt{n^2+1}}{2n}.
\end{cases} \tag{14}
$$

$$
\begin{cases}
u = \dfrac{\sqrt{l^2+1}-l-1}{2}, \\
v = -\dfrac{\sqrt{m^2+1}+m+1}{2}, \\
w = \dfrac{\sqrt{n^2+1}-n-1}{2}, \\
x = \dfrac{l+m+n+1+\sqrt{l^2+1}+\sqrt{m^2+1}-\sqrt{n^2+1}}{2l}, \\
y = \dfrac{l+m+n+1-\sqrt{l^2+1}-\sqrt{m^2+1}-\sqrt{n^2+1}}{2m}, \\
z = \dfrac{l+m+n+1-\sqrt{l^2+1}+\sqrt{m^2+1}+\sqrt{n^2+1}}{2n}.
\end{cases} \tag{15}
$$

$$
\begin{cases}
u = \dfrac{\sqrt{l^2+1}-l-1}{2}, \\
v = \dfrac{\sqrt{m^2+1}-m-1}{2}, \\
w = -\dfrac{\sqrt{n^2+1}+n+1}{2}, \\
x = \dfrac{l+m+n+1+\sqrt{l^2+1}-\sqrt{m^2+1}+\sqrt{n^2+1}}{2l}, \\
y = \dfrac{l+m+n+1-\sqrt{l^2+1}+\sqrt{m^2+1}+\sqrt{n^2+1}}{2m}, \\
z = \dfrac{l+m+n+1-\sqrt{l^2+1}-\sqrt{m^2+1}-\sqrt{n^2+1}}{2n}.
\end{cases} \tag{16}
$$

Define $\alpha, \beta, \gamma \in (0, \pi/2)$ and σ by

$$
\sin^2\alpha = \frac{a}{s}, \qquad \sin^2\beta = \frac{b}{s}, \qquad \sin^2\gamma = \frac{c}{s}, \qquad \sigma = \frac{\alpha+\beta+\gamma}{2}.
$$

The fourth equation in (9) corresponds to a value of r_1 as follows.

$$
r_1 = \frac{r\bigl(l+m+n-1+\sqrt{l^2+1}-\sqrt{m^2+1}-\sqrt{n^2+1}\bigr)}{2l}.
$$

Since

$$
r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},
$$

$$
l = \frac{s-a}{r} = \frac{s}{r_A}, \qquad m = \frac{s-b}{r} = \frac{s}{r_B}, \qquad n = \frac{s-c}{r} = \frac{s}{r_C},
$$

it holds that

$$
\begin{aligned}
&\frac{r\big(l+m+n-1+\sqrt{l^2+1}-\sqrt{m^2+1}-\sqrt{n^2+1}\big)}{2l}\\
&=\frac{r_A}{2}\left(1-\sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}}+\sqrt{\frac{(s-a)bc}{s^3}}-\sqrt{\frac{a(s-b)c}{s^3}}-\sqrt{\frac{ab(s-c)}{s^3}}\right)\\
&=\frac{r_A}{2}(1-\cos\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma-\sin\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\cos\gamma)\\
&=\frac{r_A(1-\cos(\beta+\gamma-\alpha))}{2}\\
&=r_A\sin^2(\sigma-\alpha).
\end{aligned}
$$

By making similar calculations on every last three equations in (9), (10), (11), (12), (13), (14), (15) and (16), we obtain the following respective solutions in Case 1.

$$
\begin{cases} r_1=r_A\sin^2(\sigma-\alpha),\\ r_2=r_B\sin^2(\sigma-\beta),\\ r_3=r_C\sin^2(\sigma-\gamma).\end{cases} \tag{17}
$$

$$
\begin{cases} r_1=r_A\sin^2\sigma,\\ r_2=r_B\sin^2(\sigma-\gamma),\\ r_3=r_C\sin^2(\sigma-\beta).\end{cases} \tag{18}
$$

$$
\begin{cases} r_1=r_A\sin^2(\sigma-\gamma),\\ r_2=r_B\sin^2\sigma,\\ r_3=r_C\sin^2(\sigma-\alpha).\end{cases} \tag{19}
$$

$$
\begin{cases} r_1=r_A\sin^2(\sigma-\beta),\\ r_2=r_B\sin^2(\sigma-\alpha),\\ r_3=r_C\sin^2\sigma.\end{cases} \tag{20}
$$

$$
\begin{cases} r_1=r_A\cos^2(\sigma-\alpha),\\ r_2=r_B\cos^2(\sigma-\beta),\\ r_3=r_C\cos^2(\sigma-\gamma).\end{cases} \tag{21}
$$

$$
\begin{cases} r_1=r_A\cos^2\sigma,\\ r_2=r_B\cos^2(\sigma-\gamma),\\ r_3=r_C\cos^2(\sigma-\beta).\end{cases} \tag{22}
$$

$$
\begin{cases} r_1=r_A\cos^2(\sigma-\gamma),\\ r_2=r_B\cos^2\sigma,\\ r_3=r_C\cos^2(\sigma-\alpha).\end{cases} \tag{23}
$$

$$
\begin{cases} r_1=r_A\cos^2(\sigma-\beta),\\ r_2=r_B\cos^2(\sigma-\alpha),\\ r_3=r_C\cos^2\sigma.\end{cases} \tag{24}
$$

4.2 Cases 2 & 3

In Case 2, we have that the following three disjunctions of equations hold.

$$\begin{aligned} BD_2 + D_3C + D_2D_3 &= BD_A + D_AC \quad \text{or} \quad BD_2 + D_3C - D_2D_3 = BD_A + D_AC, \\ AE_1 - CE_3 + E_1E_3 &= AE_A - CE_A \quad \text{or} \quad AE_1 - CE_3 - E_1E_3 = AE_A - CE_A, \\ AF_1 - BF_2 + F_1F_2 &= AF_A - BF_A \quad \text{or} \quad AF_1 - BF_2 - F_1F_2 = AF_A - BF_A. \end{aligned}$$

By expressing the lengths by the radii and the angle sizes, we obtain from the first disjunction that

$$r_2 \tan\frac{B}{2} + r_3 \tan\frac{C}{2} + 2\sqrt{r_2r_3} = r_A \tan\frac{B}{2} + r_A \tan\frac{C}{2} \tag{25}$$

or

$$r_2 \tan\frac{B}{2} + r_3 \tan\frac{C}{2} - 2\sqrt{r_2r_3} = \ rA \tan\frac{B}{2} + r_A \tan\frac{C}{2}, \tag{26}$$

we obtain from the second disjunction that

$$r_1 \cot\frac{A}{2} - r_3 \tan\frac{C}{2} + 2\sqrt{r_1r_3} = r_A \cot\frac{A}{2} - r_A \tan\frac{C}{2} \tag{27}$$

or

$$r_1 \cot\frac{A}{2} - r_3 \tan\frac{C}{2} - 2\sqrt{r_1r_3} = r_A \cot\frac{A}{2} - r_A \tan\frac{C}{2}, \tag{28}$$

and we obtain from the third disjunction that

$$r_1 \cot\frac{A}{2} - r_2 \tan\frac{B}{2} + 2\sqrt{r_1r_2} = r_A \cot\frac{A}{2} - r_A \tan\frac{B}{2} \tag{29}$$

or

$$r_1 \cot\frac{A}{2} - r_2 \tan\frac{B}{2} - 2\sqrt{r_1r_2} = r_A \cot\frac{A}{2} - r_A \tan\frac{B}{2}. \tag{30}$$

Define l, $\bar{m}$, $\bar{n}$ by

$$l = \cot\frac{A}{2}, \qquad \bar{m} = \tan\frac{B}{2}, \qquad \bar{n} = \tan\frac{C}{2}. \tag{31}$$

Define u, v, w, x, y, z by

$$u = \begin{cases} -\dfrac{\sqrt{r_2r_3}}{r_A} & \text{if (25) holds,} \\ \dfrac{\sqrt{r_2r_3}}{r_A} & \text{if (26) holds,} \end{cases}$$

$$v = \begin{cases} \dfrac{\sqrt{r_1r_3}}{r_A} & \text{if (27) holds,} \\ -\dfrac{\sqrt{r_1r_3}}{r_A} & \text{if (28) holds,} \end{cases}$$

$$w = \begin{cases} \dfrac{\sqrt{r_1r_2}}{r_A} & \text{if (29) holds,} \\ -\dfrac{\sqrt{r_1r_2}}{r_A} & \text{if (30) holds,} \end{cases}$$

$$x = \frac{r_1}{r_A}, \qquad y = \frac{r_2}{r_A}, \qquad z = \frac{r_3}{r_A}.$$

Then we have

$$\begin{cases} \bar{m}y + \bar{n}z - 2u = \bar{m} + \bar{n}, \\ lx - \bar{n}z + 2v = l - \bar{n}, \\ lx - \bar{m}y + 2w = l - \bar{m}, \\ xy = w^2, \\ xz = v^2, \\ yz = u^2. \end{cases} \tag{32}$$

In Case 3, we have

$$\begin{gathered} -BD_2 - D_3C + D_2D_3 = BD_A + D_AC, \\ -AE_1 + CE_3 + E_1E_3 = AE_A - CE_A \text{ or } -AE_1 + CE_3 - E_1E_3 = AE_A - CE_A, \\ -AF_1 + BF_2 + F_1F_2 = AF_A - BF_A \text{ or } -AF_1 + BF_2 - F_1F_2 = AF_A - BF_A. \end{gathered}$$

By expressing the lengths by the radii and the angle sizes, we obtain from the first equation that

$$-r_2 \tan\frac{B}{2} - r_3 \tan\frac{C}{2} + 2\sqrt{r_2r_3} = r_A \tan\frac{B}{2} + r_A \tan\frac{C}{2}, \tag{33}$$

we obtain form the second conjunction that

$$-r_1 \cot\frac{A}{2} + r_3 \tan\frac{C}{2} + 2\sqrt{r_1r_3} = r_A \cot\frac{A}{2} - r_A \tan\frac{C}{2} \tag{34}$$

or

$$-r_1 \cot\frac{A}{2} + r_3 \tan\frac{C}{2} - 2\sqrt{r_1r_3} = r_A \cot\frac{A}{2} - r_A \tan\frac{C}{2}, \tag{35}$$

and we obtain from the third conjunction that

$$-r_1 \cot\frac{A}{2} + r_2 \tan\frac{B}{2} + 2\sqrt{r_1r_2} = r_A \cot\frac{A}{2} - r_A \tan\frac{B}{2} \tag{36}$$

or

$$-r_1 \cot\frac{A}{2} + r_2 \tan\frac{B}{2} - 2\sqrt{r_1r_2} = r_A \cot\frac{A}{2} - r_A \tan\frac{B}{2}. \tag{37}$$

Define l, $\bar{m}$, $\bar{n}$ by (31). Define u, v, w, x, y, z by

$$\begin{gathered} u = -\frac{\sqrt{r_2r_3}}{r_A} \\ v = \begin{cases} \dfrac{\sqrt{r_1r_3}}{r_A} & \text{if (34) holds,} \\ -\dfrac{\sqrt{r_1r_3}}{r_A} & \text{if (35) holds,} \end{cases} \\ w = \begin{cases} \dfrac{\sqrt{r_1r_2}}{r_A} & \text{if (36) holds,} \\ -\dfrac{\sqrt{r_1r_2}}{r_A} & \text{if (37) holds,} \end{cases} \\ x = -\frac{r_1}{r_A}, \qquad y = -\frac{r_2}{r_A}, \qquad z = -\frac{r_3}{r_A}. \end{gathered}$$

Then we have the same system of equations as (32).

For any triangle ABC, if l, $\bar{m}$, $\bar{n}$ are defined by (31), then $l\bar{m}\bar{n} = l - \bar{m} - \bar{n}$ holds. On the other hand, if positive reals l, $\bar{m}$, $\bar{n}$ satisfy $l\bar{m}\bar{n} = l - \bar{m} - \bar{n}$, then there exists a triangle ABC that satisfies (31). Thus, Case 2 and Case 3 can be unified and reduced into solving the system of equations (32) for u, v, w, x, y, z with positive real parameters l, $\bar{m}$, $\bar{n}$ under the restriction $l\bar{m}\bar{n} = l - \bar{m} - \bar{n}$.

As we will show in Appendix A. the system of equations has the following 8 solutions.

$$\begin{cases} u = \dfrac{\sqrt{l^2+1} - l - 1}{2}, \\ v = \dfrac{\sqrt{\bar{m}^2+1} + \bar{m} - 1}{2}, \\ w = \dfrac{\sqrt{\bar{n}^2+1} + \bar{n} - 1}{2}, \\ x = \dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} + 1}{2l}, \\ y = \dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} - 1}{2\bar{m}}, \\ z = \dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} - 1}{2\bar{n}}. \end{cases} \tag{38}$$

$$\begin{cases} u = \dfrac{\sqrt{l^2+1} - l - 1}{2}, \\ v = -\dfrac{\sqrt{\bar{m}^2+1} - \bar{m} + 1}{2}, \\ w = -\dfrac{\sqrt{\bar{n}^2+1} - \bar{n} + 1}{2}, \\ x = \dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} + 1}{2l}, \\ y = \dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} - 1}{2\bar{m}}, \\ z = \dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} - 1}{2\bar{n}}. \end{cases} \tag{39}$$

$$\begin{cases} u = \dfrac{\sqrt{l^2+1} - l + 1}{2}, \\ v = -\dfrac{\sqrt{\bar{m}^2+1} - \bar{m} - 1}{2}, \\ w = \dfrac{\sqrt{\bar{n}^2+1} + \bar{n} + 1}{2}, \\ x = \dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} - 1}{2l}, \\ y = \dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} + 1}{2\bar{m}}, \\ z = \dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} + 1}{2\bar{n}}. \end{cases} \tag{40}$$

$$\begin{cases} u = \dfrac{\sqrt{l^2+1} - l + 1}{2}, \\ v = \dfrac{\sqrt{\bar{m}^2+1} + \bar{m} + 1}{2}, \\ w = -\dfrac{\sqrt{\bar{n}^2+1} - \bar{n} - 1}{2}, \\ x = \dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} - 1}{2l}, \\ y = \dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} + 1}{2\bar{m}}, \\ z = \dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} + 1}{2\bar{n}}. \end{cases} \tag{41}$$

$$\begin{cases} u = -\dfrac{\sqrt{l^2+1} + l - 1}{2}, \\ v = -\dfrac{\sqrt{\bar{m}^2+1} - \bar{m} - 1}{2}, \\ w = -\dfrac{\sqrt{\bar{n}^2+1} - \bar{n} - 1}{2}, \\ x = -\dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} + 1}{2l}, \\ y = -\dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} - 1}{2\bar{m}}, \\ z = -\dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} - 1}{2\bar{n}}. \end{cases} \tag{42}$$

$$\begin{cases} u = -\dfrac{\sqrt{l^2+1} + l - 1}{2}, \\ v = \dfrac{\sqrt{\bar{m}^2+1} + \bar{m} + 1}{2}, \\ w = \dfrac{\sqrt{\bar{n}^2+1} + \bar{n} + 1}{2}, \\ x = -\dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} - l + \bar{m} + \bar{n} + 1}{2l}, \\ y = -\dfrac{\sqrt{l^2+1} + \sqrt{\bar{m}^2+1} - \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} - 1}{2\bar{m}}, \\ z = -\dfrac{\sqrt{l^2+1} - \sqrt{\bar{m}^2+1} + \sqrt{\bar{n}^2+1} + l - \bar{m} - \bar{n} - 1}{2\bar{n}}. \end{cases} \tag{43}$$

$$\begin{cases} u = -\dfrac{\sqrt{l^2+1}+l+1}{2}, \\ v = \dfrac{\sqrt{\bar{m}^2+1}+\bar{m}-1}{2}, \\ w = -\dfrac{\sqrt{\bar{n}^2+1}-\bar{n}+1}{2}, \\ x = -\dfrac{\sqrt{l^2+1}+\sqrt{\bar{m}^2+1}-\sqrt{\bar{n}^2+1}-l+\bar{m}+\bar{n}-1}{2l}, \\ y = -\dfrac{\sqrt{l^2+1}+\sqrt{\bar{m}^2+1}+\sqrt{\bar{n}^2+1}+l-\bar{m}-\bar{n}+1}{2\bar{m}}, \\ z = -\dfrac{\sqrt{l^2+1}-\sqrt{\bar{m}^2+1}-\sqrt{\bar{n}^2+1}+l-\bar{m}-\bar{n}+1}{2\bar{n}}. \end{cases} \tag{44}$$

$$\begin{cases} u = -\dfrac{\sqrt{l^2+1}+l+1}{2}, \\ v = -\dfrac{\sqrt{\bar{m}^2+1}-\bar{m}+1}{2}, \\ w = \dfrac{\sqrt{\bar{n}^2+1}+\bar{n}-1}{2}, \\ x = -\dfrac{\sqrt{l^2+1}-\sqrt{\bar{m}^2+1}+\sqrt{\bar{n}^2+1}-l+\bar{m}+\bar{n}-1}{2l}, \\ y = -\dfrac{\sqrt{l^2+1}-\sqrt{\bar{m}^2+1}-\sqrt{\bar{n}^2+1}+l-\bar{m}-\bar{n}+1}{2\bar{m}}, \\ z = -\dfrac{\sqrt{l^2+1}+\sqrt{\bar{m}^2+1}+\sqrt{\bar{n}^2+1}+l-\bar{m}-\bar{n}+1}{2\bar{n}}. \end{cases} \tag{45}$$

Define $\alpha_A, \beta_A, \gamma_A \in (0, +\infty)$ and σ_A by

$$\sinh^2 \alpha_A = \frac{a}{s-a}, \qquad \sinh^2 \beta_A = \frac{s-c}{s-a}, \qquad \sinh^2 \gamma_A = \frac{s-b}{s-a},$$
$$\sigma_A = \frac{\alpha_A + \beta_A + \gamma_A}{2}.$$

The fourth equation in (38) corresponds to a value of r_1 as follows.

$$r_1 = \frac{r_A\left(\sqrt{l^2+1}-\sqrt{\bar{m}^2+1}-\sqrt{\bar{n}^2+1}+l-\bar{m}-\bar{n}+1\right)}{2l}$$

Since

$$r_A = \sqrt{\frac{s(s-b)(s-c)}{s-a}},$$
$$l = \frac{s}{r_A} = \frac{s-a}{r}, \qquad \bar{m} = \frac{s-c}{r_A} = \frac{s-a}{r_C}, \qquad \bar{n} = \frac{s-b}{r_A} = \frac{s-a}{r_B},$$

it holds that

$$
\begin{aligned}
&\frac{r_A\left(\sqrt{l^2+1}-\sqrt{\bar{m}^2+1}-\sqrt{\bar{n}^2+1}+l-\bar{m}-\bar{n}+1\right)}{2l}\\
&=\frac{r}{2}\left(\sqrt{\frac{bcs}{(s-a)^3}}-\sqrt{\frac{ab(s-b)}{(s-a)^3}}-\sqrt{\frac{ac(s-c)}{(s-a)^3}}+\sqrt{\frac{s(s-b)(s-c)}{(s-a)^3}}+1\right)\\
&=\frac{r}{2}(\cosh\alpha_A\cosh\beta_A\cosh\gamma_A-\sinh\alpha_A\cosh\beta_A\sinh\gamma_A\\
&\qquad-\sinh\alpha_A\sinh\beta_A\cosh\gamma_A+\cosh\alpha_A\sinh\beta_A\sinh\gamma_A+1)\\
&=\frac{r_A(\cosh(\beta_A+\gamma_A-\alpha_A)+1)}{2}\\
&=r_A\sinh^2(\sigma_A-\alpha_A).
\end{aligned}
$$

By making similar calculations on every last three equations in (38), (39), (40), (41), (42), (43), (44) and (45), we obtain the following respective solutions in Cases 2 and 3.

$$
\begin{cases}
r_1=r\cosh^2(\sigma_A-\alpha_A),\\
r_2=r_C\sinh^2(\sigma_A-\beta_A),\\
r_3=r_B\sinh^2(\sigma_A-\gamma_A).
\end{cases}
\tag{46}
$$

$$
\begin{cases}
r_1=r\cosh^2\sigma_A,\\
r_2=r_C\sinh^2(\sigma_A-\gamma_A),\\
r_3=r_B\sinh^2(\sigma_A-\beta_A).
\end{cases}
\tag{47}
$$

$$
\begin{cases}
r_1=r\cosh^2(\sigma_A-\gamma_A),\\
r_2=r_C\sinh^2\sigma_A,\\
r_3=r_B\sinh^2(\sigma_A-\alpha_A).
\end{cases}
\tag{48}
$$

$$
\begin{cases}
r_1=r\cosh^2(\sigma_A-\beta_A),\\
r_2=r_C\sinh^2(\sigma_A-\alpha_A),\\
r_3=r_B\sinh^2\sigma_A.
\end{cases}
\tag{49}
$$

$$
\begin{cases}
r_1=r\sinh^2(\sigma_A-\alpha_A),\\
r_2=r_C\cosh^2(\sigma_A-\beta_A),\\
r_3=r_B\cosh^2(\sigma_A-\gamma_A).
\end{cases}
\tag{50}
$$

$$
\begin{cases}
r_1=r\sinh^2\sigma_A,\\
r_2=r_C\cosh^2(\sigma_A-\gamma_A),\\
r_3=r_B\cosh^2(\sigma_A-\beta_A).
\end{cases}
\tag{51}
$$

$$
\begin{cases}
r_1=r\sinh^2(\sigma_A-\gamma_A),\\
r_2=r_C\cosh^2\sigma_A,\\
r_3=r_B\cosh^2(\sigma_A-\alpha_A).
\end{cases}
\tag{52}
$$

$$
\begin{cases}
r_1=r\sinh^2(\sigma_A-\beta_A),\\
r_2=r_C\cosh^2(\sigma_A-\alpha_A),\\
r_3=r_B\cosh^2\sigma_A.
\end{cases}
\tag{53}
$$

4.3 Cases 4 & 5

Define $\alpha_B, \beta_B, \gamma_B \in (0, +\infty)$ and σ_B by

$$\sinh^2 \alpha_B = \frac{s-c}{s-b}, \qquad \sinh^2 \beta_B = \frac{b}{s-b}, \qquad \sinh^2 \gamma_B = \frac{s-a}{s-b},$$
$$\sigma_B = \frac{\alpha_B + \beta_B + \gamma_B}{2}.$$

Analogously to 4.2, we obtain the following solutions in Cases 4 and 5.

$$\begin{cases} r_1 = r_C \sinh^2(\sigma_B - \alpha_B), \\ r_2 = r \cosh^2(\sigma_B - \beta_B), \\ r_3 = r_A \sinh^2(\sigma_B - \gamma_B). \end{cases} \tag{54}$$

$$\begin{cases} r_1 = r_C \sinh^2 \sigma_B, \\ r_2 = r \cosh^2(\sigma_B - \gamma_B), \\ r_3 = r_A \sinh^2(\sigma_B - \beta_B). \end{cases} \tag{55}$$

$$\begin{cases} r_1 = r_C \sinh^2(\sigma_B - \gamma_B), \\ r_2 = r \cosh^2 \sigma_B, \\ r_3 = r_A \sinh^2(\sigma_B - \alpha_B). \end{cases} \tag{56}$$

$$\begin{cases} r_1 = r_C \sinh^2(\sigma_B - \beta_B), \\ r_2 = r \cosh^2(\sigma_B - \alpha_B), \\ r_3 = r_A \sinh^2 \sigma_B. \end{cases} \tag{57}$$

$$\begin{cases} r_1 = r_C \cosh^2(\sigma_B - \alpha_B), \\ r_2 = r \sinh^2(\sigma_B - \beta_B), \\ r_3 = r_A \cosh^2(\sigma_B - \gamma_B). \end{cases} \tag{58}$$

$$\begin{cases} r_1 = r_C \cosh^2 \sigma_B, \\ r_2 = r \sinh^2(\sigma_B - \gamma_B), \\ r_3 = r_B \cosh^2(\sigma_B - \beta_B). \end{cases} \tag{59}$$

$$\begin{cases} r_1 = r_C \cosh^2(\sigma_B - \gamma_B), \\ r_2 = r \sinh^2 \sigma_B, \\ r_3 = r_A \cosh^2(\sigma_B - \alpha_B). \end{cases} \tag{60}$$

$$\begin{cases} r_1 = r_C \cosh^2(\sigma_B - \beta_B), \\ r_2 = r \sinh^2(\sigma_B - \alpha_B), \\ r_3 = r_A \cosh^2 \sigma_B. \end{cases} \tag{61}$$

4.4 Cases 6 & 7

Define $\alpha_C, \beta_C, \gamma_C \in (0, +\infty)$ and σ_C by

$$\sinh^2 \alpha_C = \frac{s-b}{s-c}, \qquad \sinh^2 \beta_C = \frac{s-a}{s-c}, \qquad \sinh^2 \gamma_C = \frac{c}{s-c},$$
$$\sigma_C = \frac{\alpha_C + \beta_C + \gamma_C}{2}.$$

Analogously to 4.2, we obtain the following solutions in Cases 6 and 7.

$$\begin{cases} r_1 = r_B \sinh^2(\sigma_C - \alpha_C), \\ r_2 = r_A \sinh^2(\sigma_C - \beta_C), \\ r_3 = r \cosh^2(\sigma_C - \gamma_C). \end{cases} \tag{62}$$

$$\begin{cases} r_1 = r_B \sinh^2 \sigma_C, \\ r_2 = r_A \sinh^2(\sigma_C - \gamma_C), \\ r_3 = r \cosh^2(\sigma_C - \beta_C). \end{cases} \tag{63}$$

$$\begin{cases} r_1 = r_B \sinh^2(\sigma_C - \gamma_C), \\ r_2 = r_A \sinh^2 \sigma_C, \\ r_3 = r \cosh^2(\sigma_C - \alpha_C). \end{cases} \tag{64}$$

$$\begin{cases} r_1 = r_B \sinh^2(\sigma_C - \beta_C), \\ r_2 = r_A \sinh^2(\sigma_C - \alpha_C), \\ r_3 = r \cosh^2 \sigma_C. \end{cases} \tag{65}$$

$$\begin{cases} r_1 = r_B \cosh^2(\sigma_C - \alpha_C), \\ r_2 = r_A \cosh^2(\sigma_C - \beta_C), \\ r_3 = r \sinh^2(\sigma_C - \gamma_C). \end{cases} \tag{66}$$

$$\begin{cases} r_1 = r_B \cosh^2 \sigma_C, \\ r_2 = r_A \cosh^2(\sigma_C - \gamma_C), \\ r_3 = r \sinh^2(\sigma_C - \beta_C). \end{cases} \tag{67}$$

$$\begin{cases} r_1 = r_B \cosh^2(\sigma_C - \gamma_C), \\ r_2 = r_A \cosh^2 \sigma_C, \\ r_3 = r \sinh^2(\sigma_C - \beta_C). \end{cases} \tag{68}$$

$$\begin{cases} r_1 = r_B \cosh^2(\sigma_C - \beta_C), \\ r_2 = r_A \cosh^2(\sigma_C - \alpha_C), \\ r_3 = r \sinh^2 \sigma_C. \end{cases} \tag{69}$$

5 Conclusion

Theorem 1. *For any triangle, there exist 32 triplets of circles such that each circle is tangent to the other two circles and to two of the sides of the reference triangle or their extensions. The radii can be expressed by* (17)–(24)*,* (46)–(53)*,* (54)–(61)*,* (62)–(69)*.*

Figures 3–34 illustrate the 32 triplets of circles for a triangle ABC such that $A = 45°$, $B = 54°$, $C = 81°$.

A Solutions of the systems of equations

In this appendix, we will solve some systems of equations by computing Gröbner bases. Although it is difficult to compute the Gröbner bases by hand, any

computer algebra system that can compute Gröbner bases should work.

Proposition 1. *The system of equations* (8) *for the variables u, v, w, x, y, z with the positive real parameters l, m, n under the restriction $lmn = l + m + n$. has* 8 *solutions* (9)–(16)*.*

Proof. Counting l, m, n among the variables in addition to u, v, w, x, y, z, we compute the reduced Gröbner basis of $\{my+nz+2u-m-n,\ lx+nz+2v-l-n,\ lx+my+2w-l-m,\ xy-w^2,\ xz-v^2,\ yz-u^2,\ lmn-l-m-n\}$ with the degree reverse lexicographical ordering $x \succ y \succ z \succ u \succ v \succ w \succ l \succ m \succ n$. The reduced Gröbner basis consists of 67 polynomials including

$$\begin{aligned}
f_1 &= (2u^2 + 2lu - 2u - l)(2u^2 + 2lu + 2u + l),\\
f_2 &= (2v^2 + 2mv - 2v - m)(2v^2 + 2mv + 2v + m),\\
f_3 &= (2w^2 + 2nw - 2w - n)(2w^2 + 2nw + 2w + n),\\
f_4 &= lx - u + v + w - l,\\
f_5 &= my + u - v + w - m,\\
f_6 &= nz + u + v - w - n.
\end{aligned}$$

By solving $\{f_1 = 0,\ f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,\ f_6 = 0\}$ for u, v, w, x, y, z, we obtain 128 solutions. Note that $lmn = l + m + n$ is equivalent to

$$l = \frac{m+n}{mn-1}. \tag{70}$$

By assigning each of the 128 solutions together with (70) to $\{my+nz+2u-m-n,\ lx+nz+2v-l-n,\ lx+my+2w-l-m,\ xy-w^2,\ xz-v^2,\ yz-u^2\}$ and then picking out the solutions such that the assignment makes all of the polynomials equal 0, we still have 8 solutions (9)–(16), which are the solutions of (8). □

Proposition 2. *The system of equations* (32) *for the variables u, v, w, x, y, z with the positive real parameters l, $\bar{m}$, $\bar{n}$ under the restriction $l\bar{m}\bar{n} = l - \bar{m} - \bar{n}$ has* 8 *solutions* (38)–(45)*.*

Proof. Counting l, $\bar{m}$, $\bar{n}$ among the variables in addition to u, v, w, x, y, z, we compute the reduced Gröbner basis of $\{\bar{m}y+\bar{n}z-2u-\bar{m}-\bar{n},\ lx-\bar{n}z+2v-l+\bar{n},\ lx-\bar{m}y+2w-l+\bar{m},\ xy-w^2,\ xz-v^2,\ yz-u^2,\ l\bar{m}\bar{n}-l+\bar{m}+\bar{n}\}$ with the degree reverse lexicographical ordering $x \succ y \succ z \succ u \succ v \succ w \succ l \succ \bar{m} \succ \bar{n}$. The reduced Gröbner basis consists of 67 polynomials including

$$\begin{aligned}
f_1 &= (2u^2 + 2lu - 2u - l)(2u^2 + 2lu + 2u + l),\\
f_2 &= (2v^2 - 2\bar{m}v - 2v + \bar{m})(2v^2 - 2\bar{m}v + 2v - \bar{m}),\\
f_3 &= (2w^2 - 2\bar{n}w - 2w + \bar{n})(2w^2 - 2\bar{n}w + 2w - \bar{n}),\\
f_4 &= lx - u + v + w - l,\\
f_5 &= \bar{m}y - u + v - w - \bar{m},\\
f_6 &= \bar{n}z - u - v + w - \bar{n}.
\end{aligned}$$

By solving $\{f_1 = 0,\ f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,\ f_6 = 0\}$ for u, v, w, x, y, z, we obtain 128 solutions. Note that $l\bar{m}\bar{n} = l - \bar{m} - \bar{n}$ is equivalent to

$$l = -\frac{\bar{m}+\bar{n}}{\bar{m}\bar{n}-1}. \tag{71}$$

By assigning each of the 128 solutions together with (71) to $\{\bar{m}y+\bar{n}z-2u-\bar{m}-\bar{n},\ lx-\bar{n}z+2v-l+\bar{n},\ lx-\bar{m}y+2w-l+\bar{m},\ xy-w^2,\ xz-v^2,\ yz-u^2\}$ and then picking out the solutions such that the assignment makes all of the polynomials equal 0, we still have 8 solutions (38)–(45), which are the solutions of (32). $\square$

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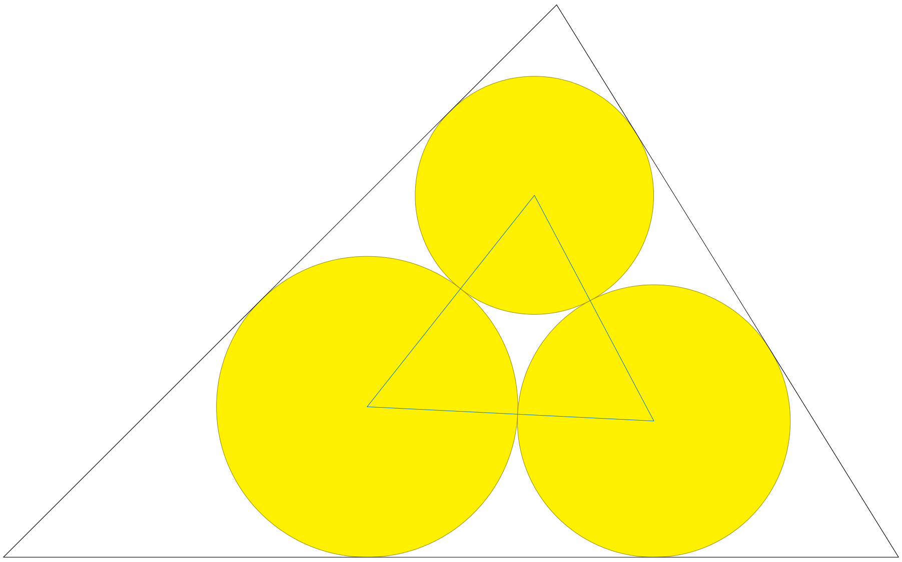
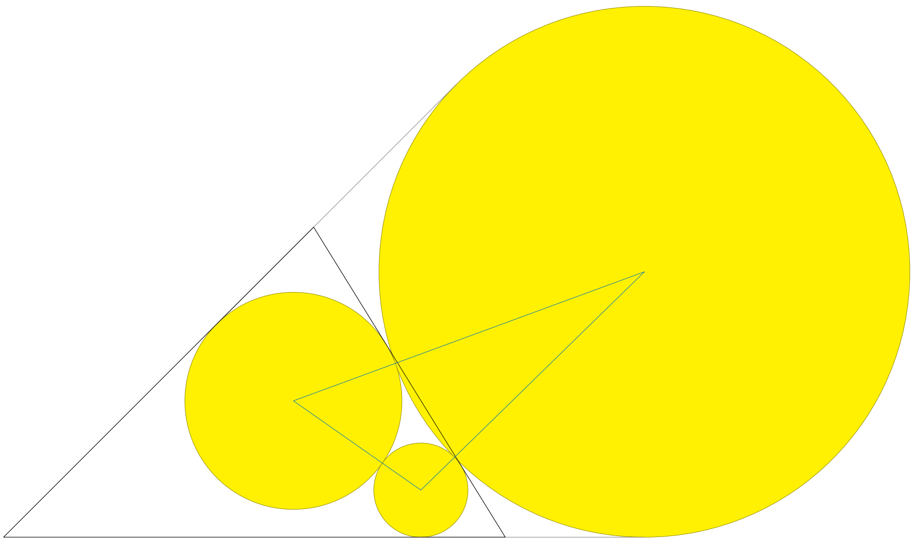


$$\text{Figure 3: } \begin{cases} r_1 = r_A \sin^2(\sigma - \alpha), \\ r_2 = r_B \sin^2(\sigma - \beta), \\ r_3 = r_C \sin^2(\sigma - \gamma). \end{cases}$$



$$\text{Figure 4: } \begin{cases} r_1 = r_A \sin^2 \sigma, \\ r_2 = r_B \sin^2(\sigma - \gamma), \\ r_3 = r_C \sin^2(\sigma - \beta). \end{cases}$$

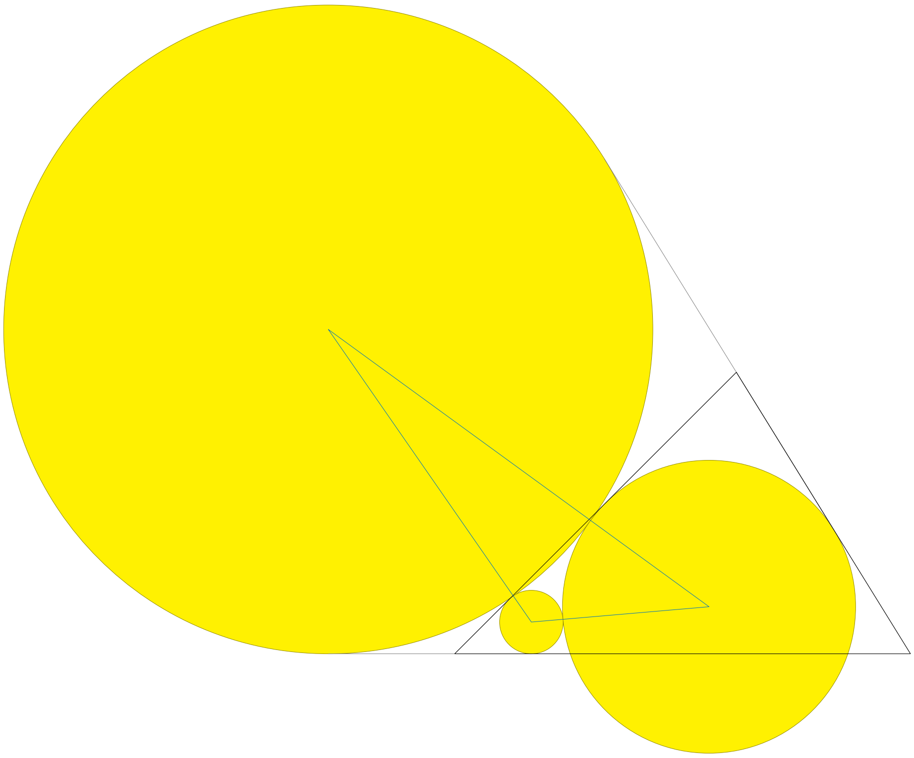


Figure 5: $\begin{cases} r_1 = r_A \sin^2(\sigma - \gamma), \\ r_2 = r_B \sin^2 \sigma, \\ r_3 = r_C \sin^2(\sigma - \alpha). \end{cases}$

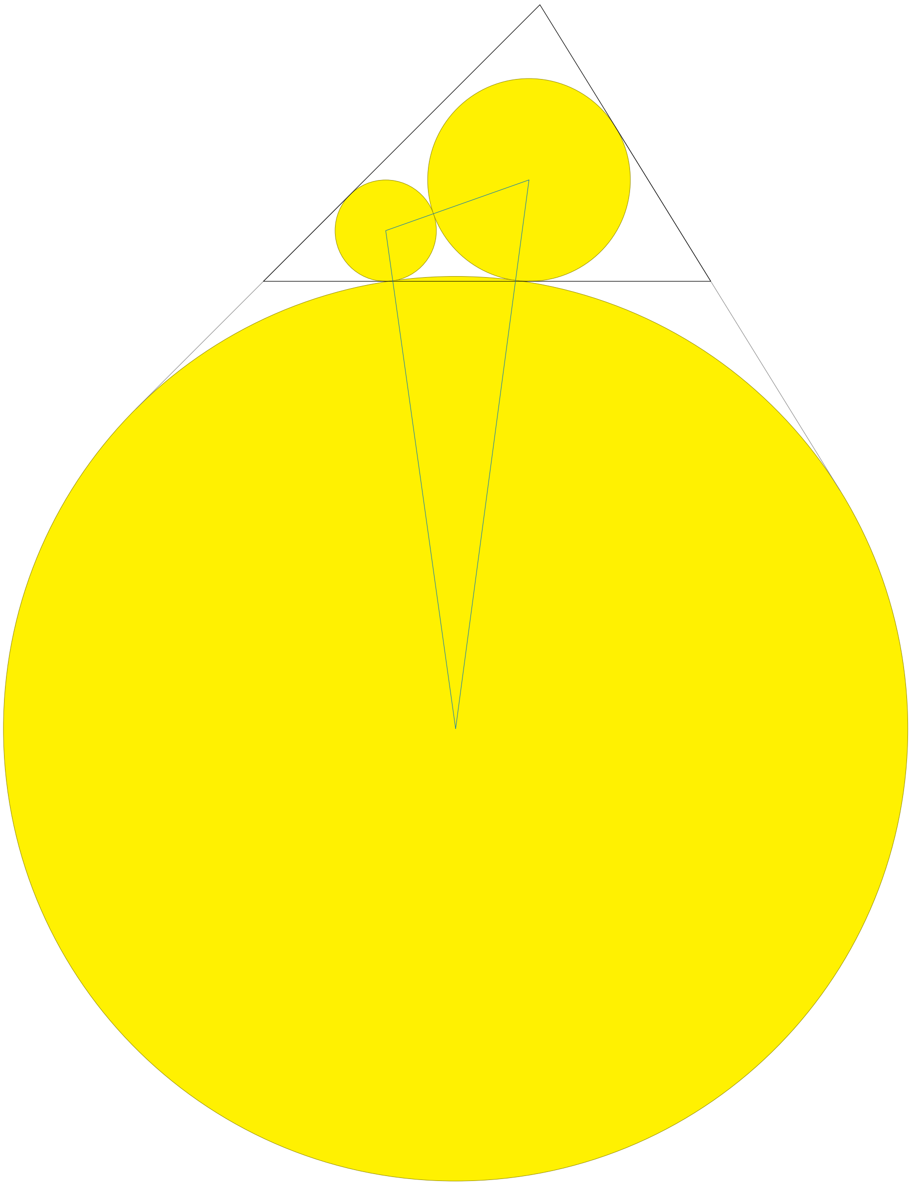


Figure 6: $\begin{cases} r_1 = r_A \sin^2(\sigma - \beta), \\ r_2 = r_B \sin^2(\sigma - \alpha), \\ r_3 = r_C \sin^2 \sigma. \end{cases}$

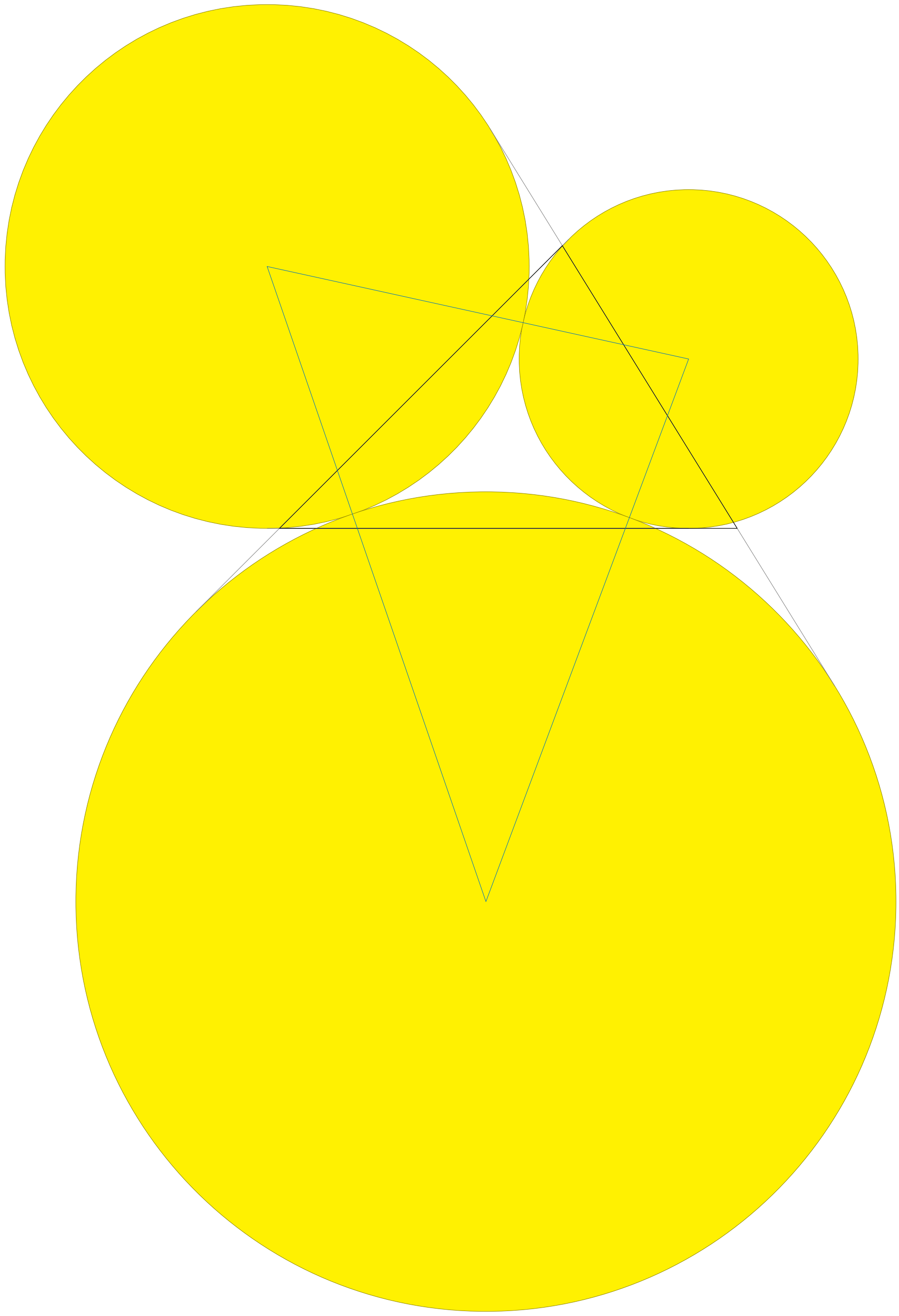


Figure 7: $\begin{cases} r_1 = r_A \cos^2(\sigma - \alpha), \\ r_2 = r_B \cos^2(\sigma - \beta), \\ r_3 = r_C \cos^2(\sigma - \gamma). \end{cases}$

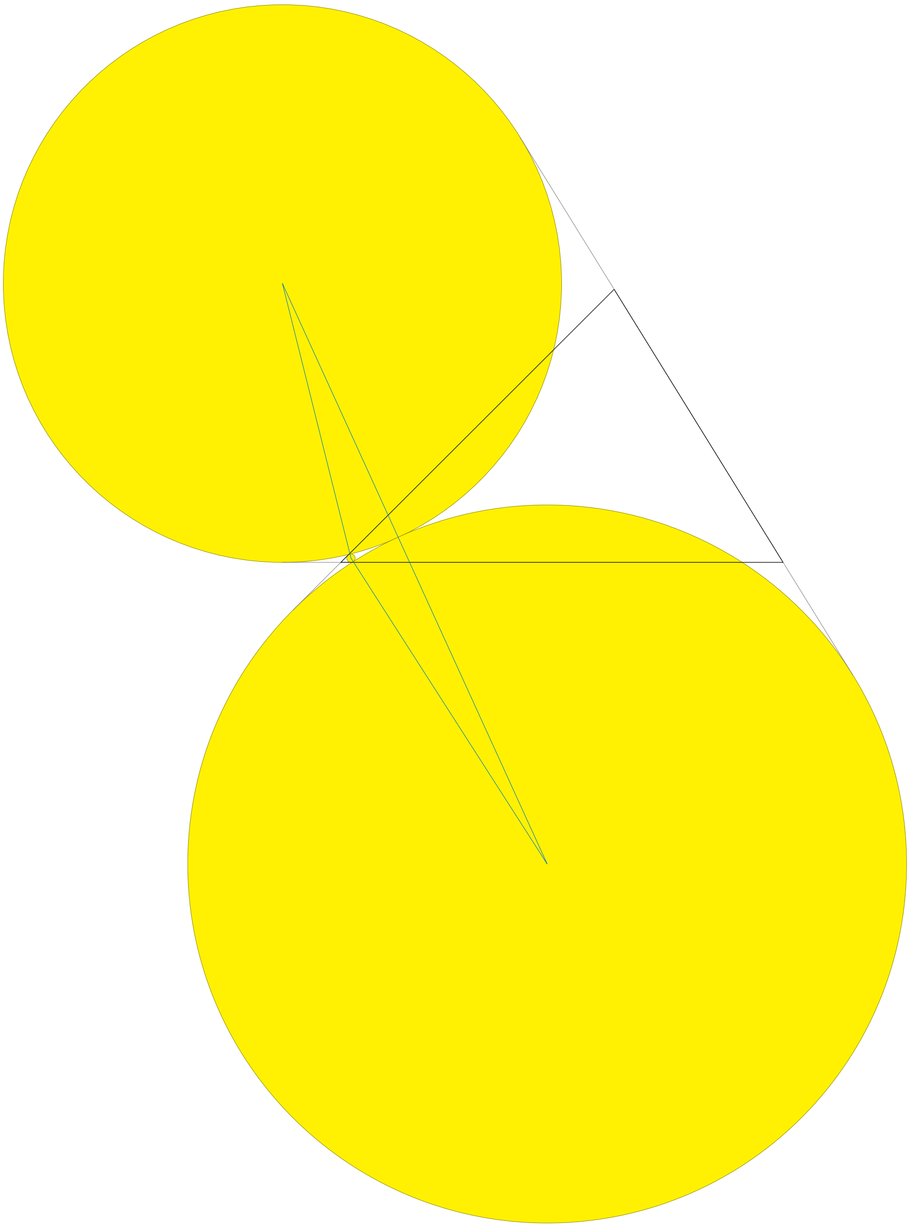


Figure 8: $\begin{cases} r_1 = r_A \cos^2 \sigma, \\ r_2 = r_B \cos^2(\sigma - \gamma), \\ r_3 = r_C \cos^2(\sigma - \beta). \end{cases}$

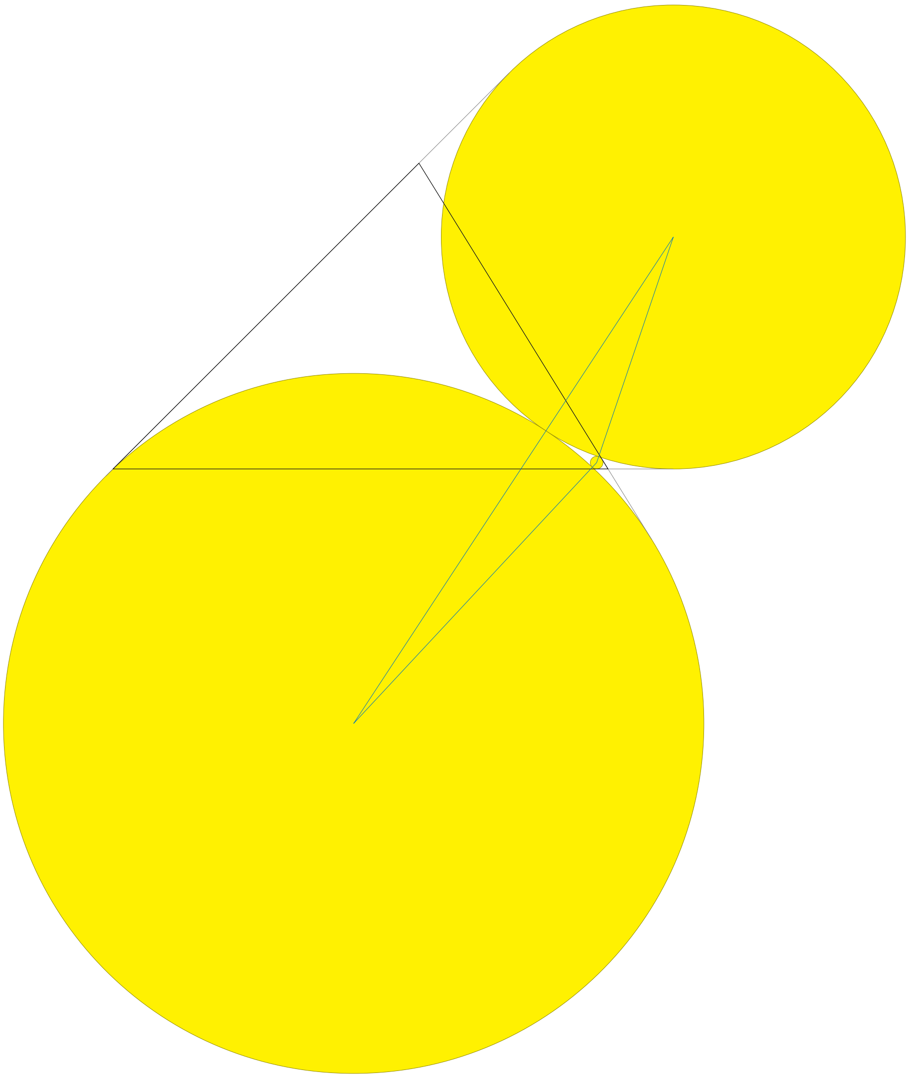


Figure 9: $\begin{cases} r_1 = r_A \cos^2(\sigma - \gamma), \\ r_2 = r_B \cos^2 \sigma, \\ r_3 = r_C \cos^2(\sigma - \alpha). \end{cases}$

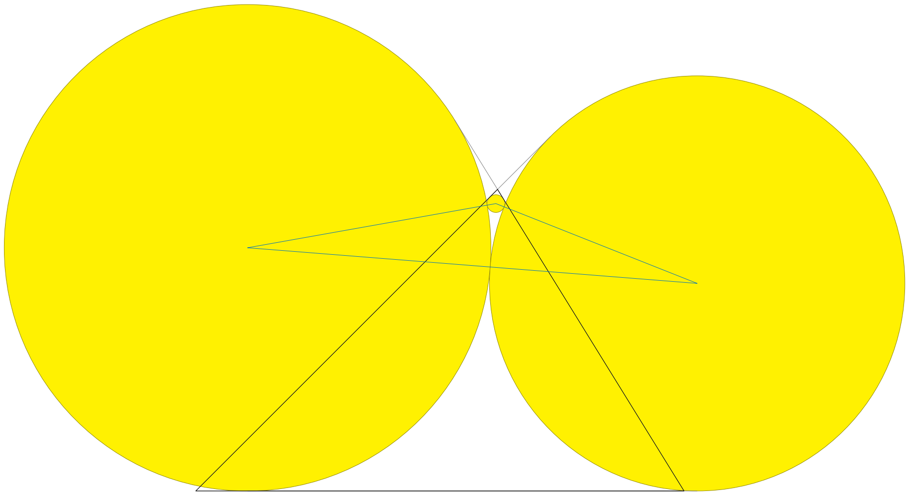


Figure 10: $\begin{cases} r_1 = r_A \cos^2(\sigma - \beta), \\ r_2 = r_B \cos^2(\sigma - \alpha), \\ r_3 = r_C \cos^2 \sigma. \end{cases}$

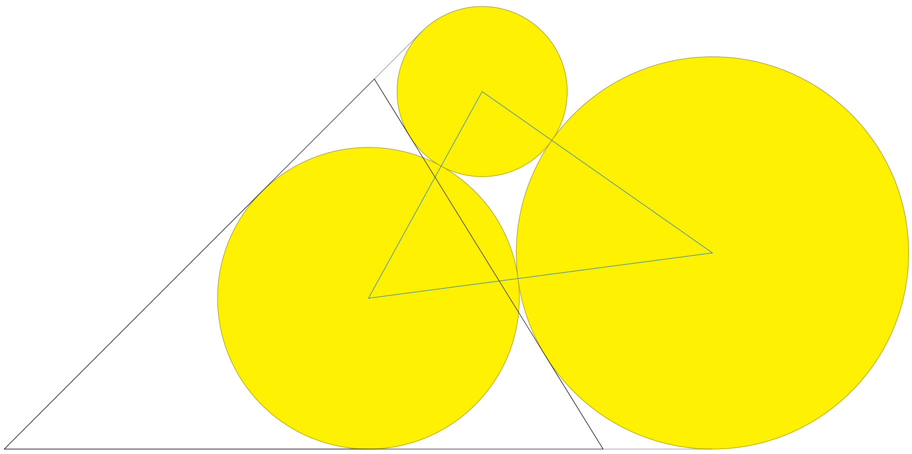


Figure 11: $\begin{cases} r_1 = r \cosh^2(\sigma_A - \alpha_A), \\ r_2 = r_C \sinh^2(\sigma_A - \beta_A), \\ r_3 = r_B \sinh^2(\sigma_A - \gamma_A). \end{cases}$

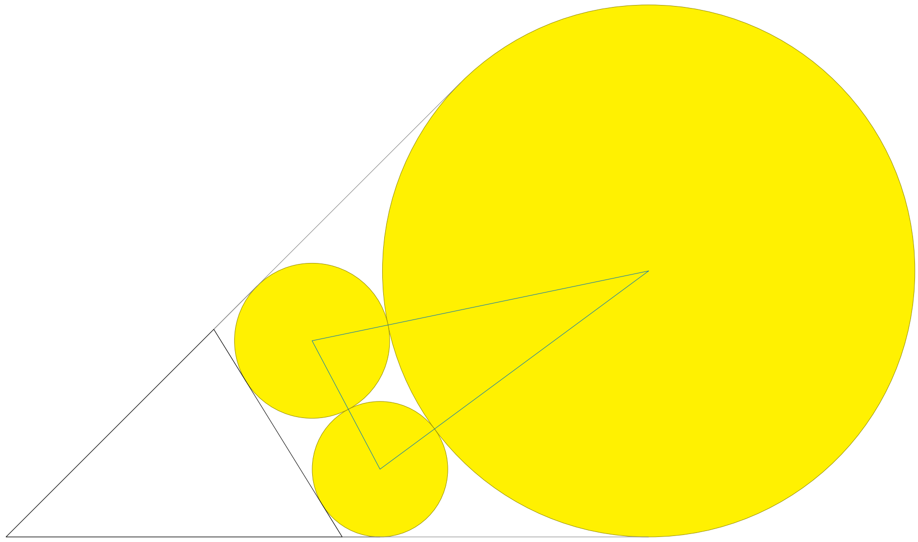


Figure 12: $\begin{cases} r_1 = r\cosh^2\sigma_A, \\ r_2 = r_C \sinh^2(\sigma_A - \gamma_A), \\ r_3 = r_B \sinh^2(\sigma_A - \beta_A). \end{cases}$

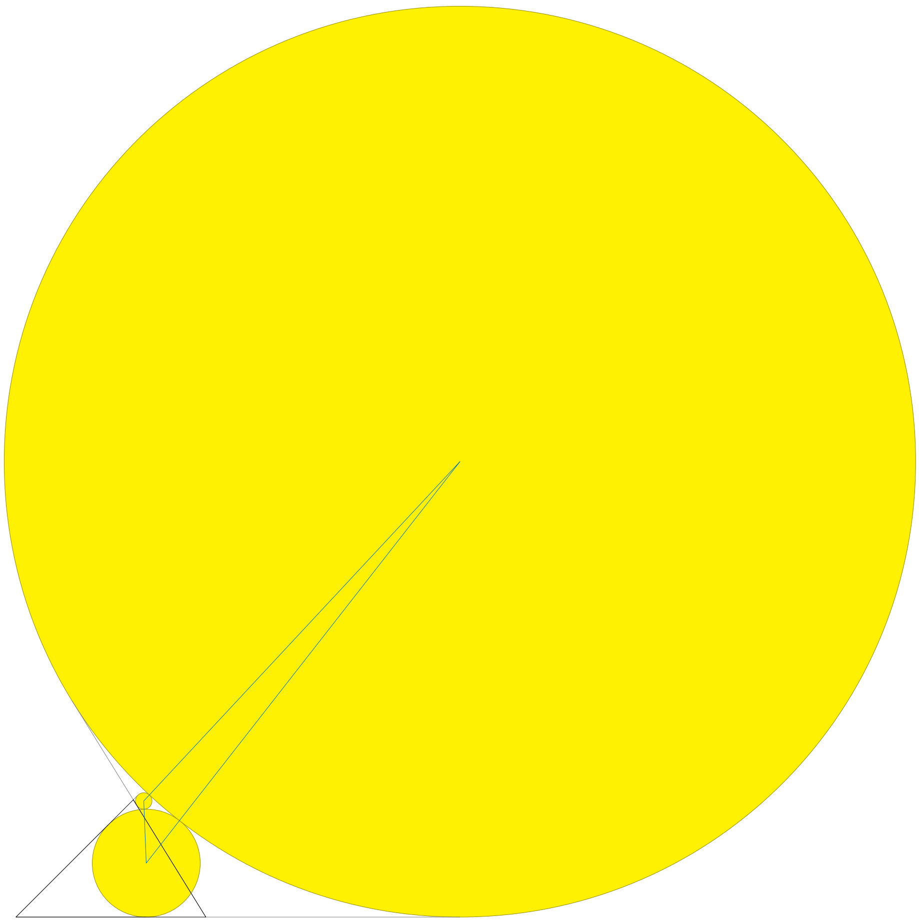


Figure 13: $\begin{cases} r_1 = r\cosh^2(\sigma_A - \gamma_A), \\ r_2 = r_C \sinh^2 \sigma_A, \\ r_3 = r_B \sinh^2(\sigma_A - \alpha_A). \end{cases}$

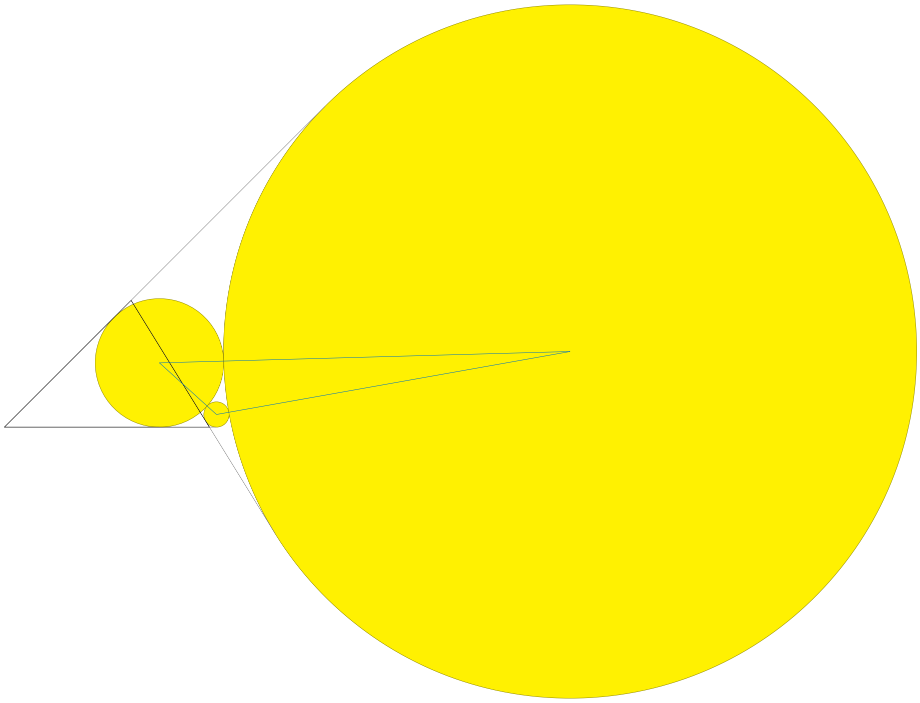


Figure 14: $\begin{cases} r_1 = r\cosh^2(\sigma_A - \beta_A), \\ r_2 = r_C \sinh^2(\sigma_A - \alpha_A), \\ r_3 = r_B \sinh^2 \sigma_A. \end{cases}$

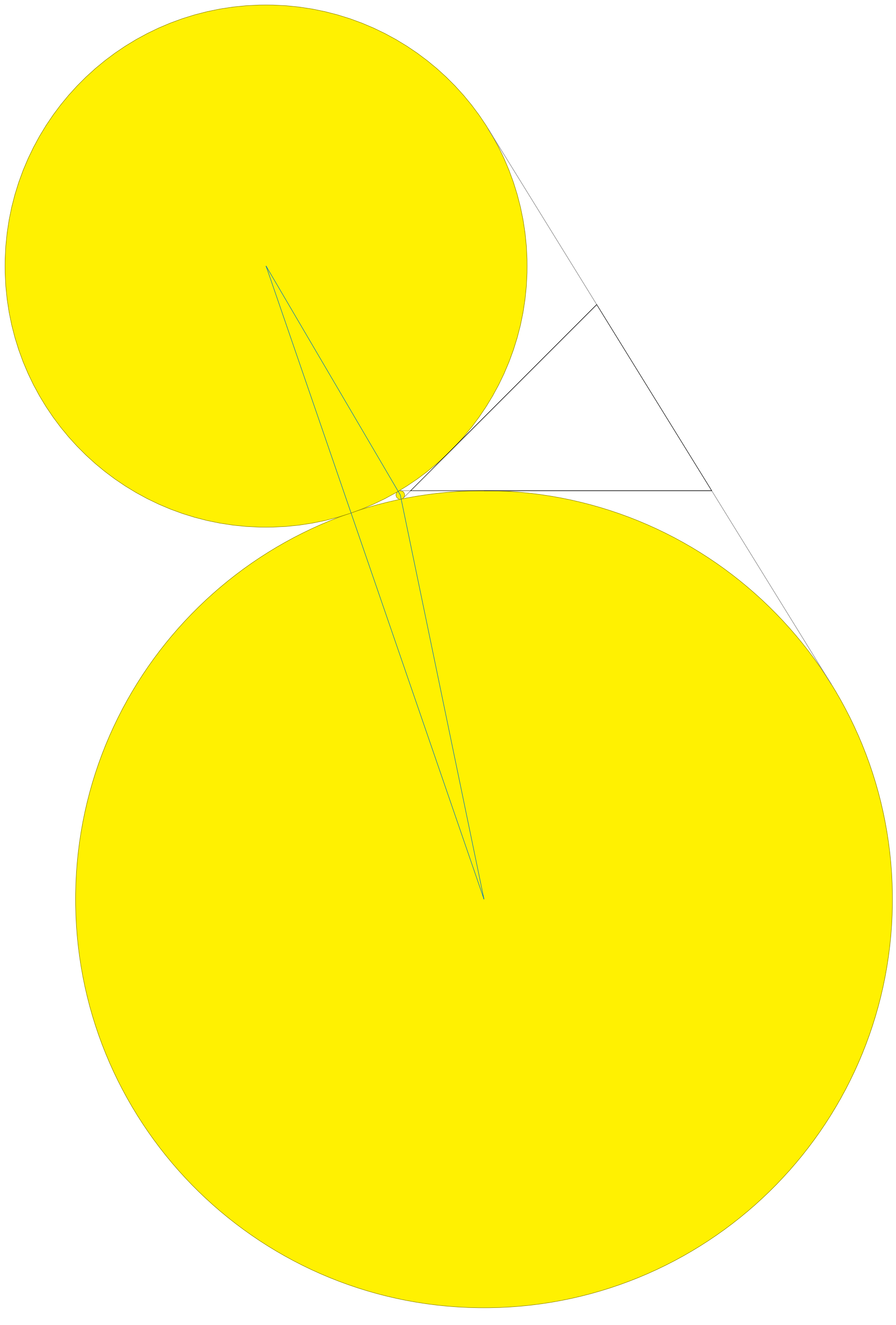


Figure 15: $\begin{cases} r_1 = r\sinh^2(\sigma_A - \alpha_A), \\ r_2 = r_C\cosh^2(\sigma_A - \beta_A), \\ r_3 = r_B\cosh^2(\sigma_A - \gamma_A). \end{cases}$

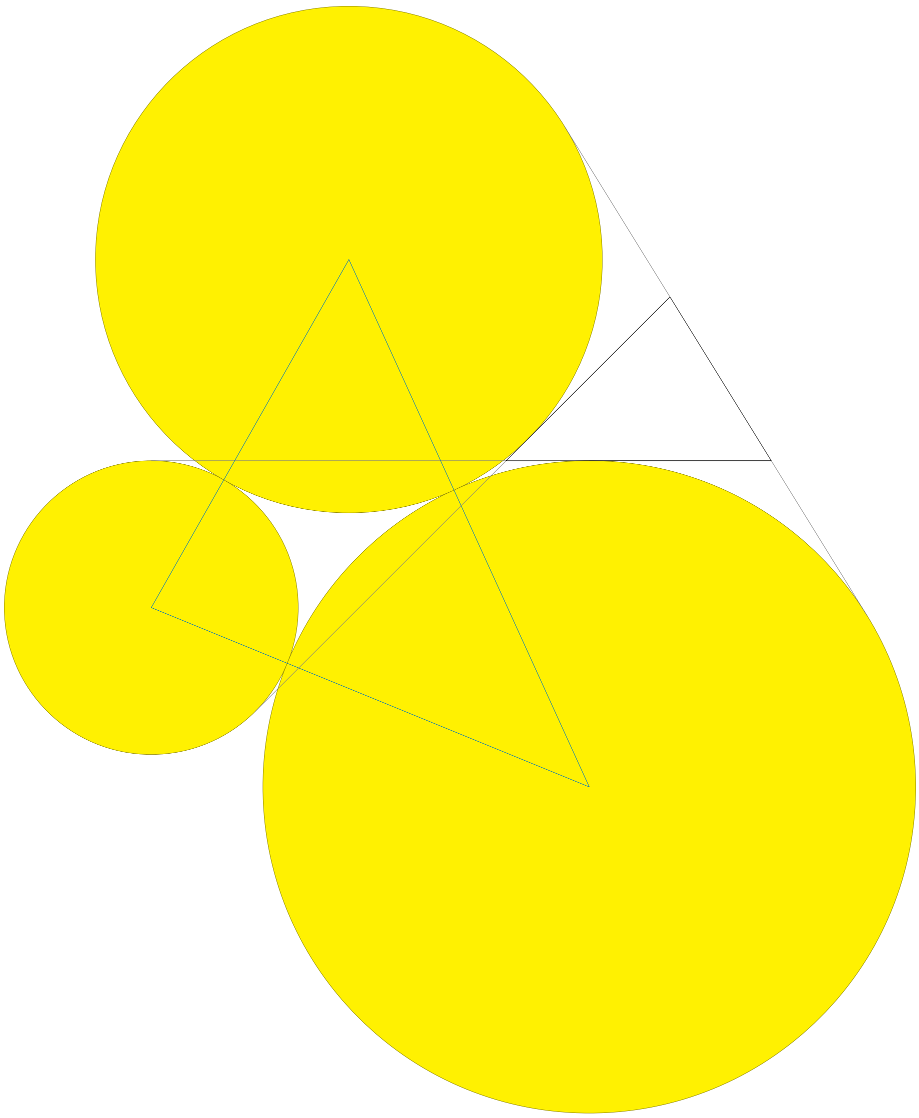


Figure 16: $\begin{cases} r_1 = r\sinh^2\sigma_A, \\ r_2 = r_C\cosh^2(\sigma_A-\gamma_A), \\ r_3 = r_B\cosh^2(\sigma_A-\beta_A). \end{cases}$

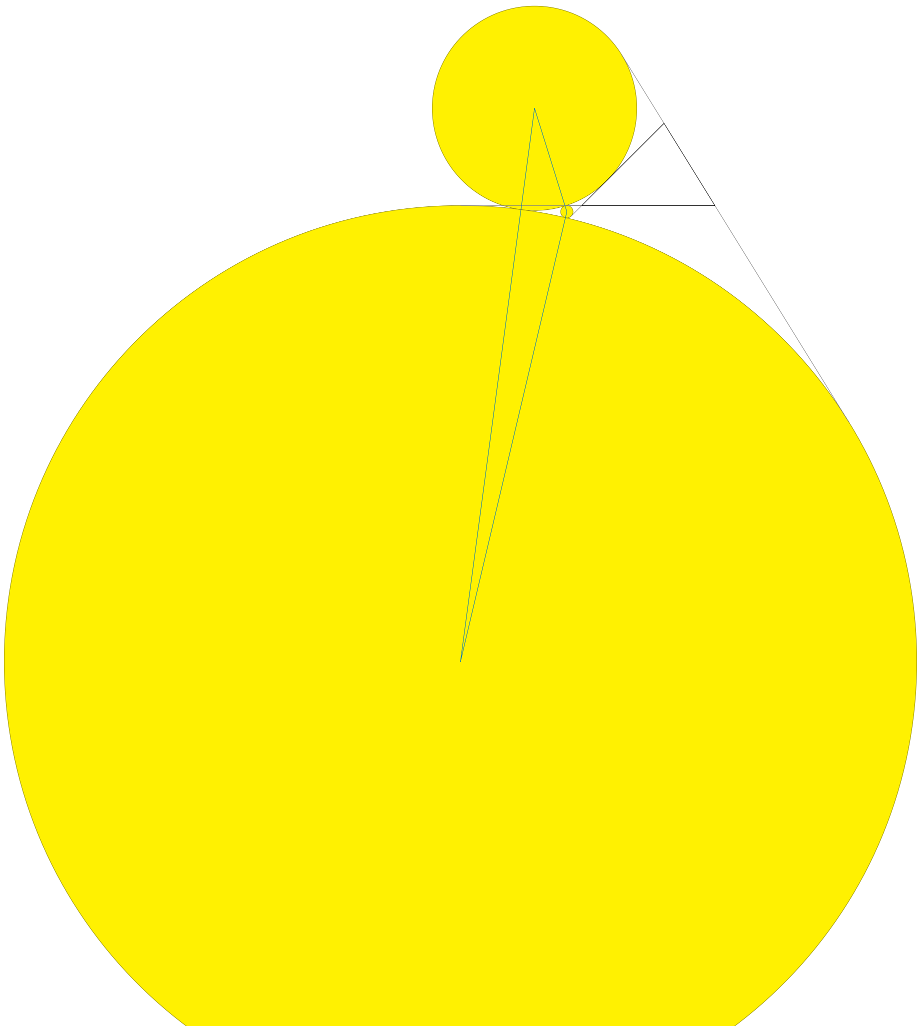


Figure 17: $\begin{cases} r_1 = r\sinh^2(\sigma_A - \gamma_A), \\ r_2 = r_C \cosh^2 \sigma_A, \\ r_3 = r_B \cosh^2(\sigma_A - \alpha_A). \end{cases}$

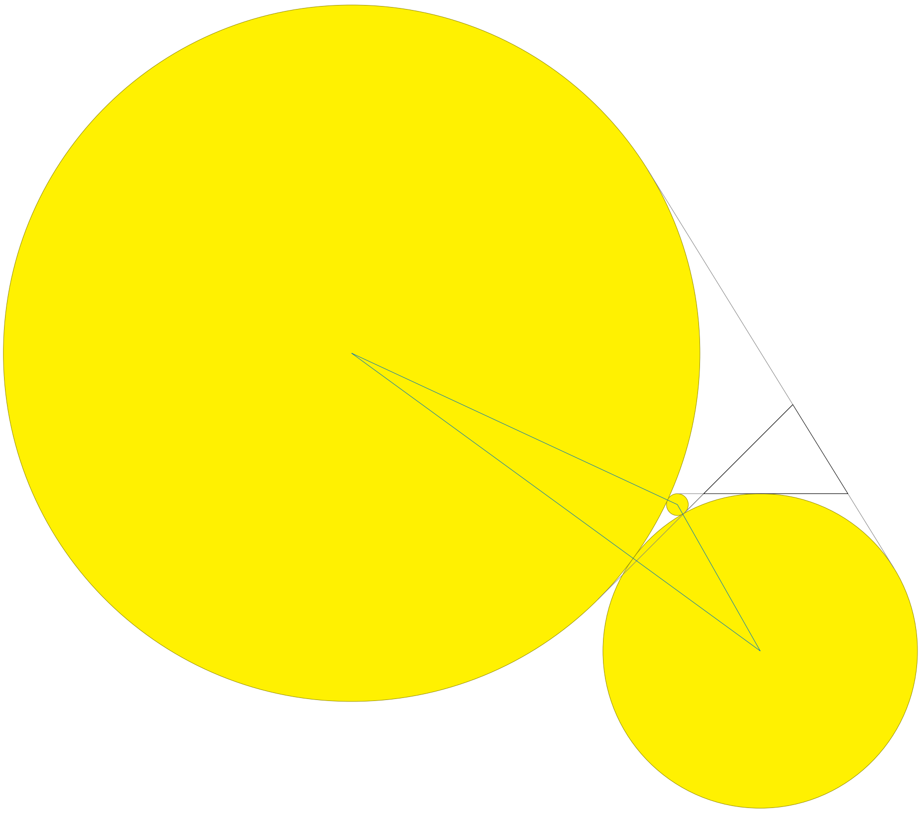


Figure 18: $\begin{cases} r_1 = r \sinh^2(\sigma_A - \beta_A), \\ r_2 = r_C \cosh^2(\sigma_A - \alpha_A), \\ r_3 = r_B \cosh^2 \sigma_A. \end{cases}$

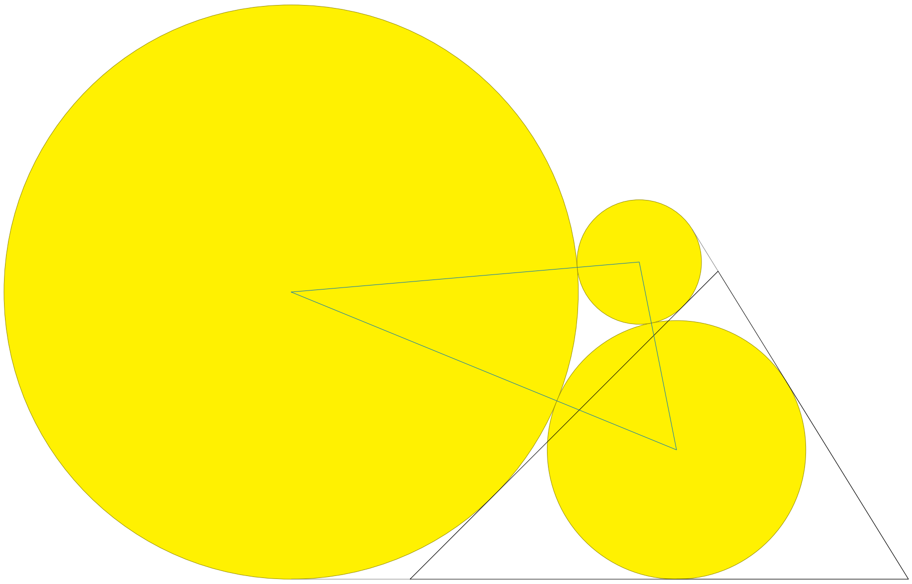


Figure 19: $\begin{cases} r_1 = r_C \sinh^2(\sigma_B - \alpha_B), \\ r_2 = r \cosh^2(\sigma_B - \beta_B), \\ r_3 = r_A \sinh^2(\sigma_B - \gamma_B). \end{cases}$

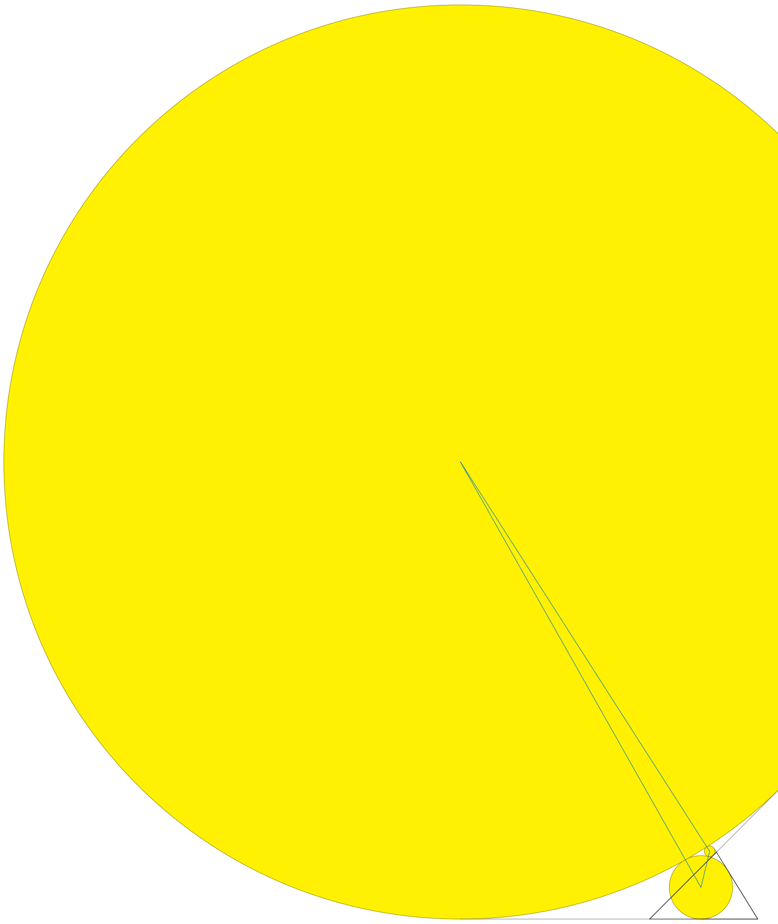


Figure 20: $\begin{cases} r_1 = r_C \sinh^2 \sigma_B, \\ r_2 = r \cosh^2(\sigma_B - \gamma_B), \\ r_3 = r_A \sinh^2(\sigma_B - \beta_B). \end{cases}$

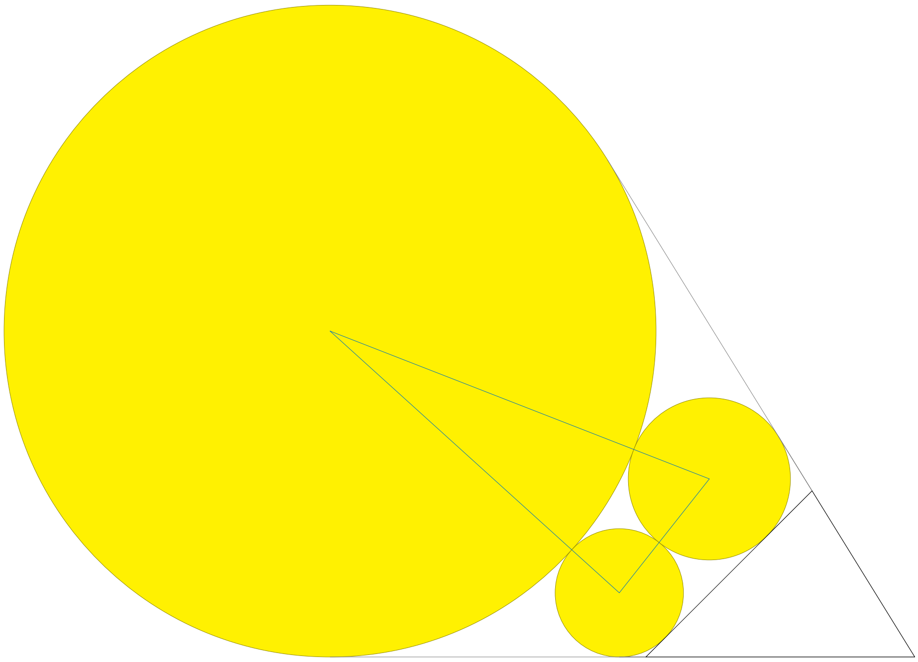


Figure 21: $\begin{cases} r_1 = r_C \sinh^2(\sigma_B - \gamma_B), \\ r_2 = r \cosh^2 \sigma_B, \\ r_3 = r_A \sinh^2(\sigma_B - \alpha_B). \end{cases}$

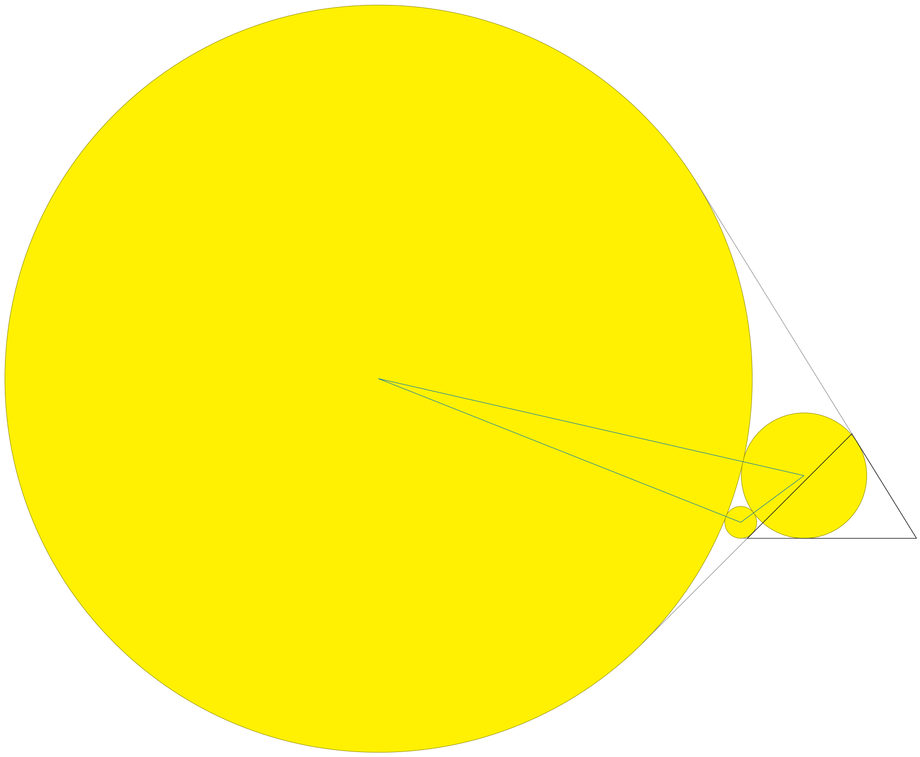


Figure 22: $\begin{cases} r_1 = r_C \sinh^2(\sigma_B - \beta_B), \\ r_2 = r \cosh^2(\sigma_B - \alpha_B), \\ r_3 = r_A \sinh^2 \sigma_B. \end{cases}$

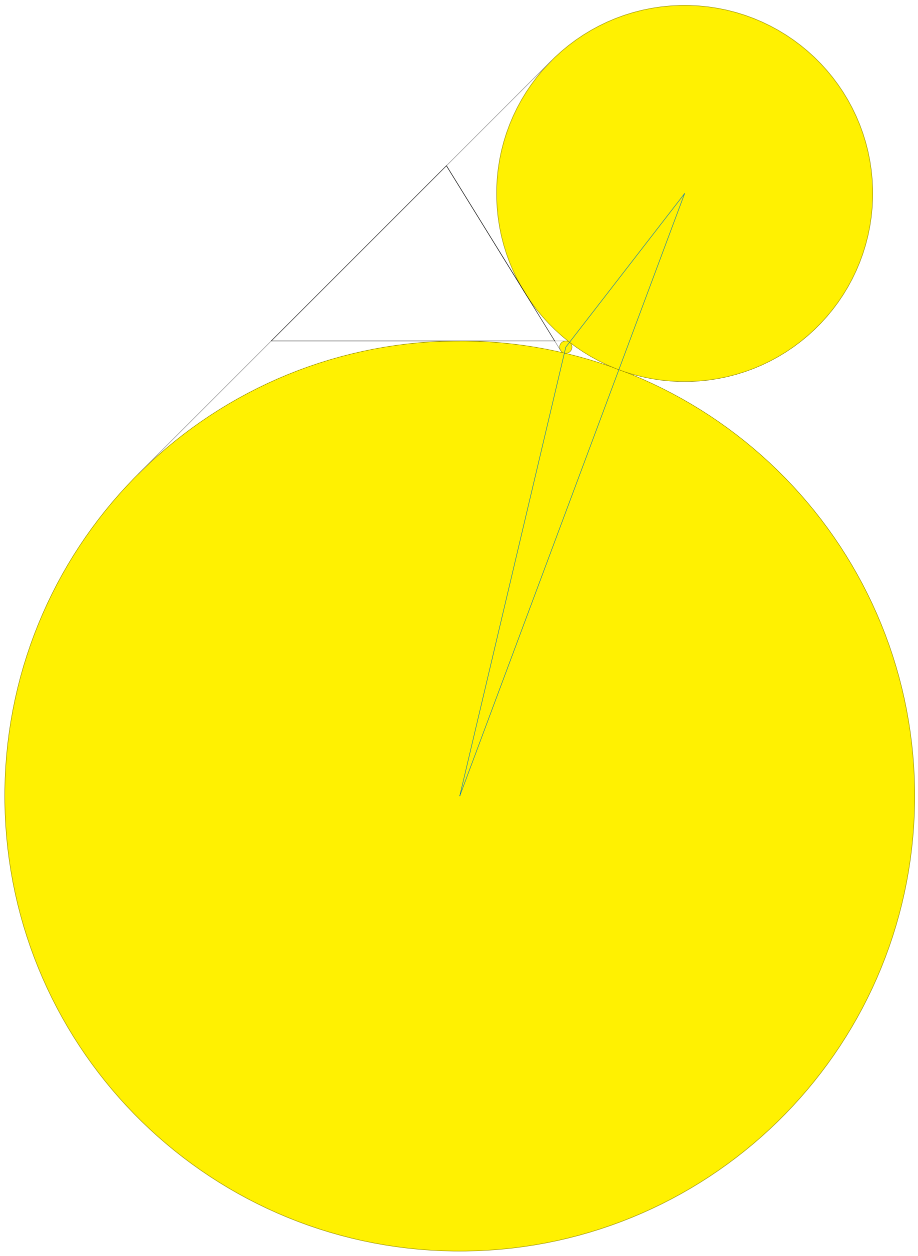


Figure 23: $\begin{cases} r_1 = r_C \cosh^2(\sigma_B - \alpha_B), \\ r_2 = r \sinh^2(\sigma_B - \beta_B), \\ r_3 = r_A \cosh^2(\sigma_B - \gamma_B). \end{cases}$

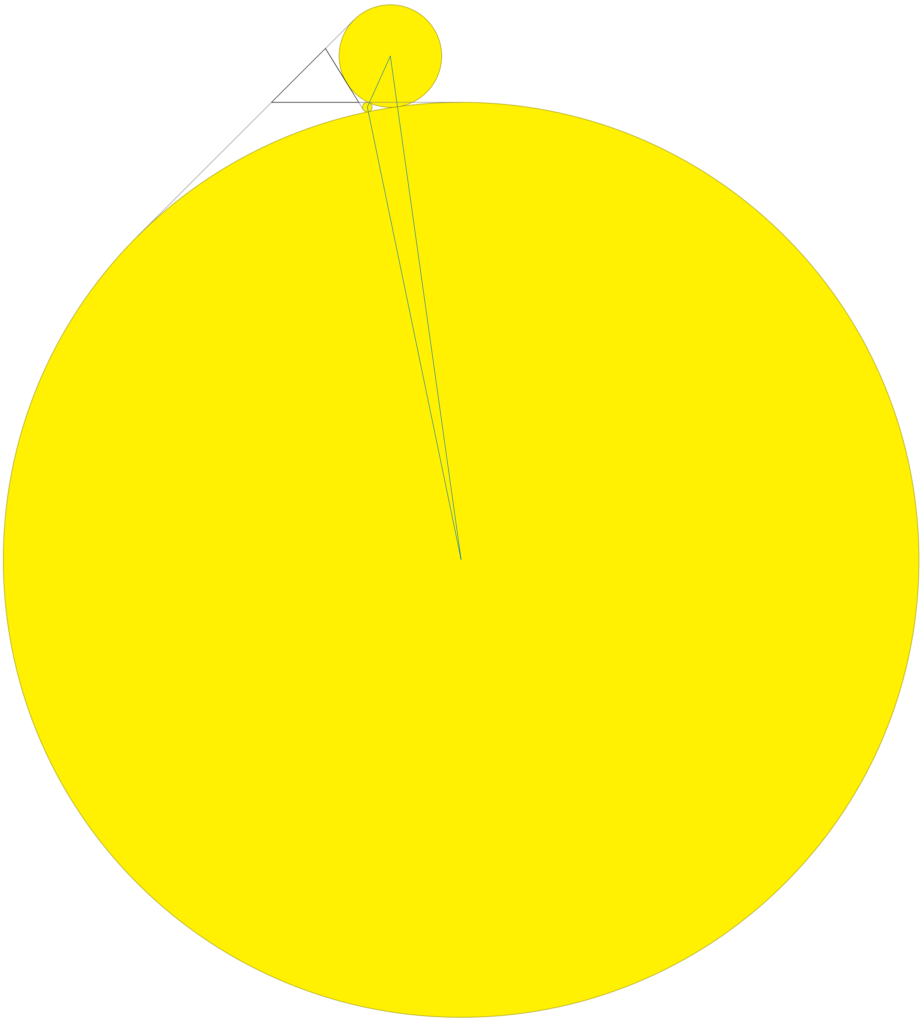


Figure 24: $\begin{cases} r_1 = r_C \cosh^2 \sigma_B, \\ r_2 = r \sinh^2(\sigma_B - \gamma_B), \\ r_3 = r_B \cosh^2(\sigma_B - \beta_B). \end{cases}$

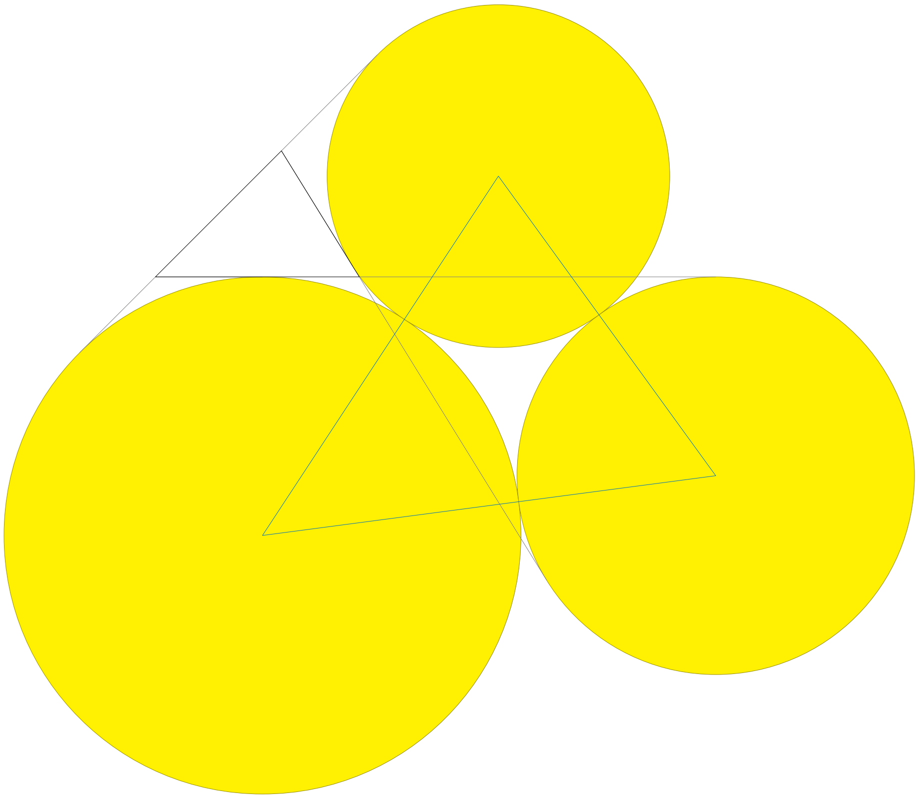


Figure 25: $\begin{cases} r_1 = r_C \cosh^2(\sigma_B - \gamma_B), \\ r_2 = r \sinh^2 \sigma_B, \\ r_3 = r_A \cosh^2(\sigma_B - \alpha_B). \end{cases}$

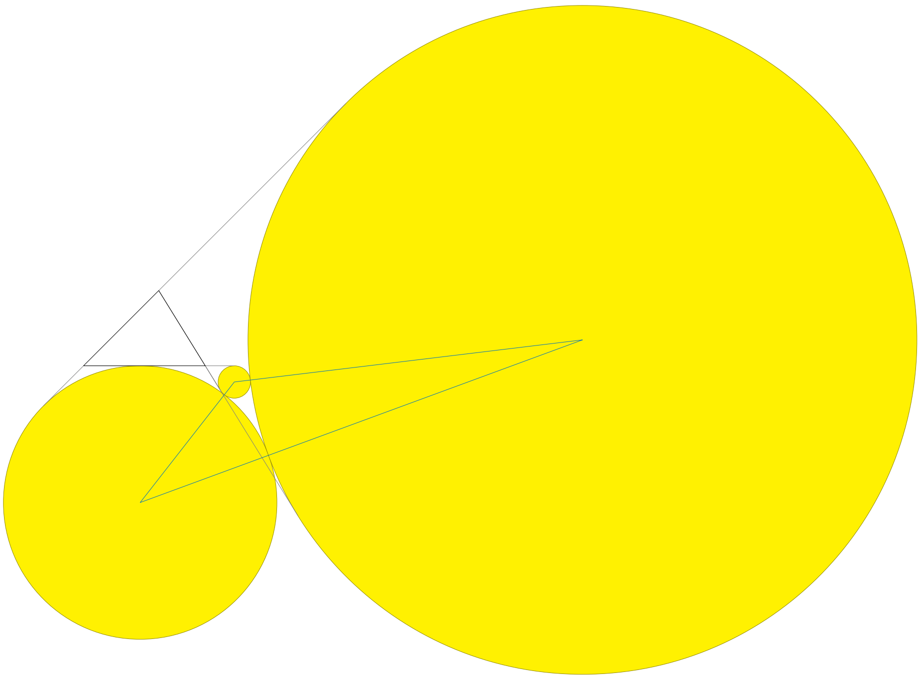


Figure 26: $\begin{cases} r_1 = r_C \cosh^2(\sigma_B - \beta_B), \\ r_2 = r \sinh^2(\sigma_B - \alpha_B), \\ r_3 = r_A \cosh^2 \sigma_B. \end{cases}$

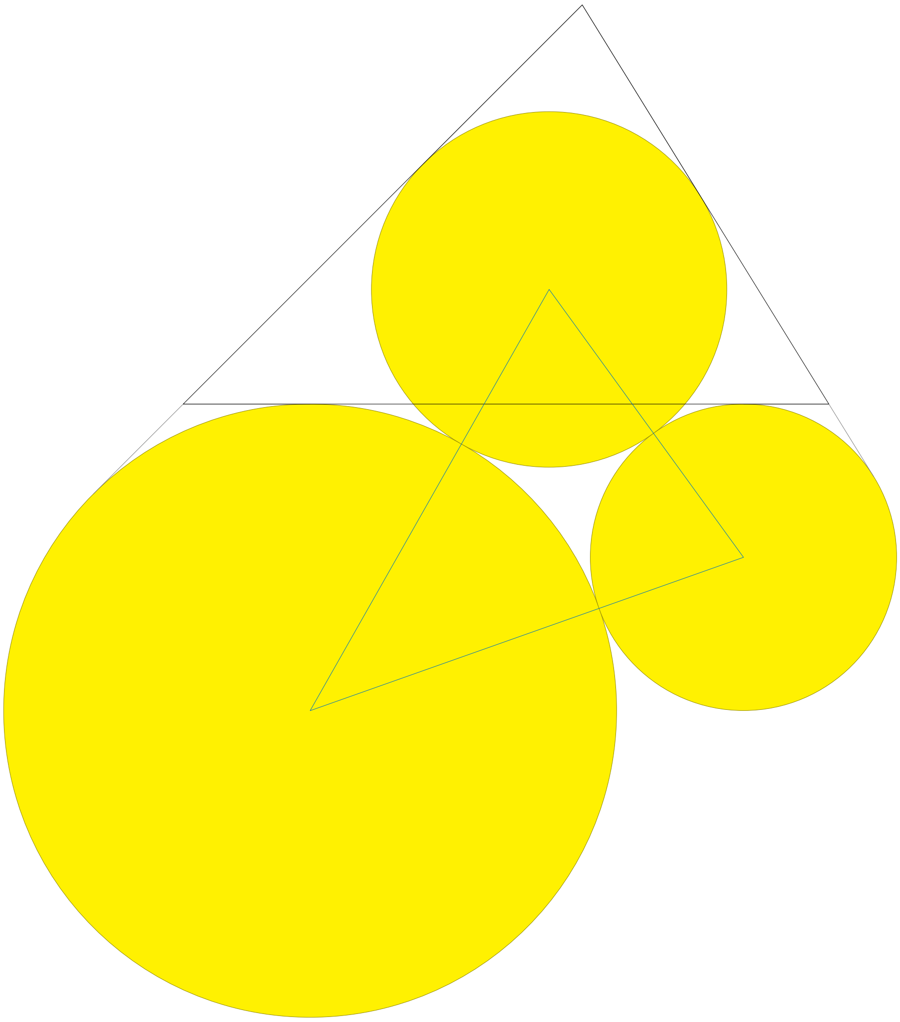


Figure 27: $$\begin{cases} r_1 = r_B \sinh^2(\sigma_C - \alpha_C), \\ r_2 = r_A \sinh^2(\sigma_C - \beta_C), \\ r_3 = r \cosh^2(\sigma_C - \gamma_C). \end{cases}$$

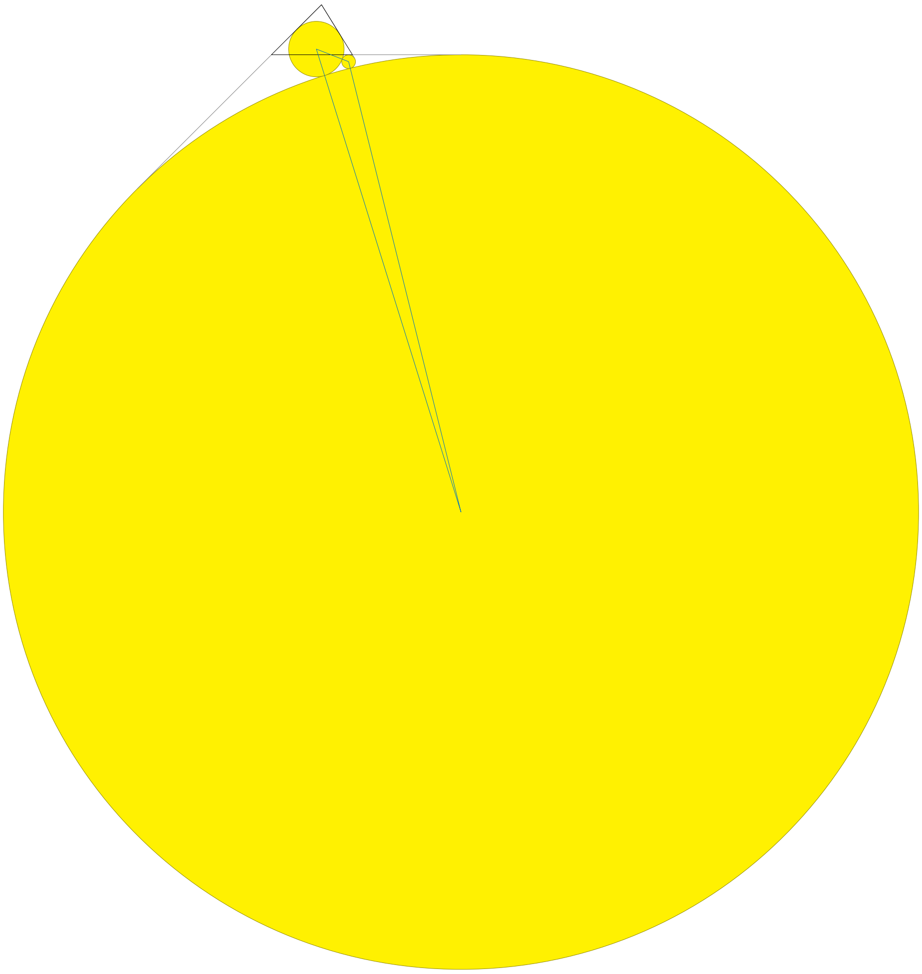


Figure 28: $\begin{cases} r_1 = r_B \sinh^2 \sigma_C, \\ r_2 = r_A \sinh^2(\sigma_C - \gamma_C), \\ r_3 = r \cosh^2(\sigma_C - \beta_C). \end{cases}$

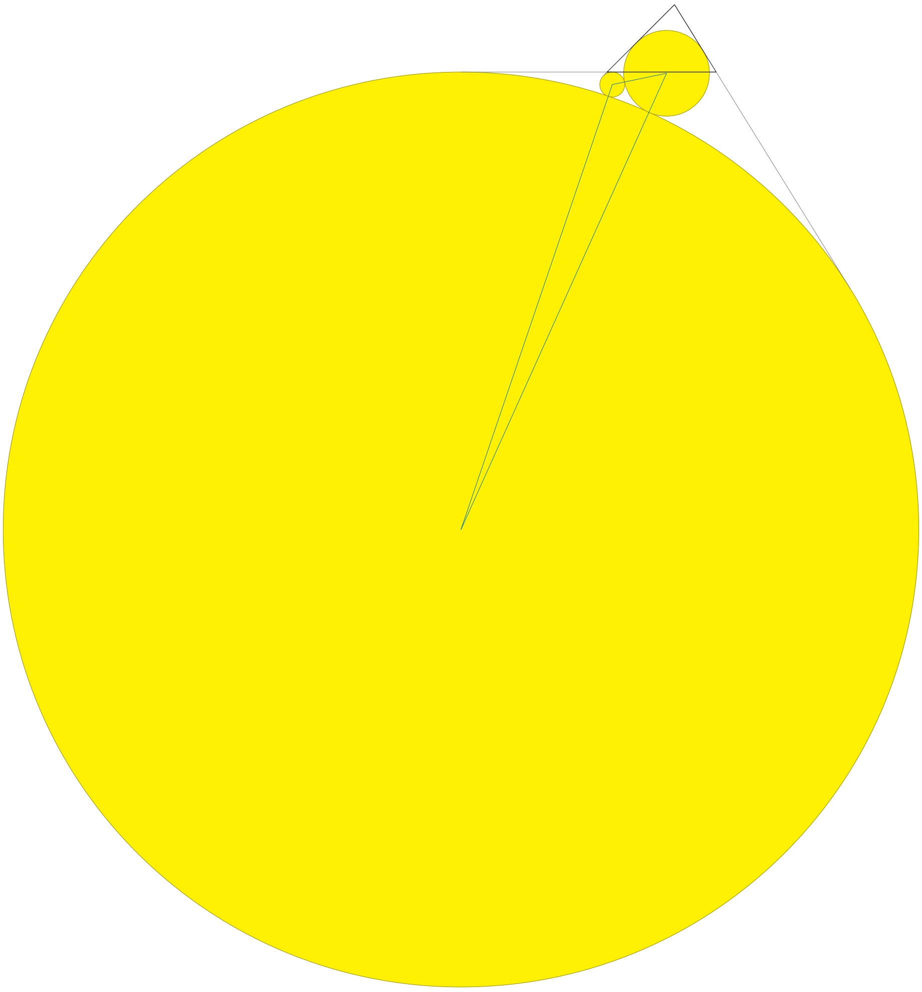


Figure 29: $\begin{cases} r_1 = r_B \sinh^2(\sigma_C - \gamma_C), \\ r_2 = r_A \sinh^2 \sigma_C, \\ r_3 = r \cosh^2(\sigma_C - \alpha_C). \end{cases}$

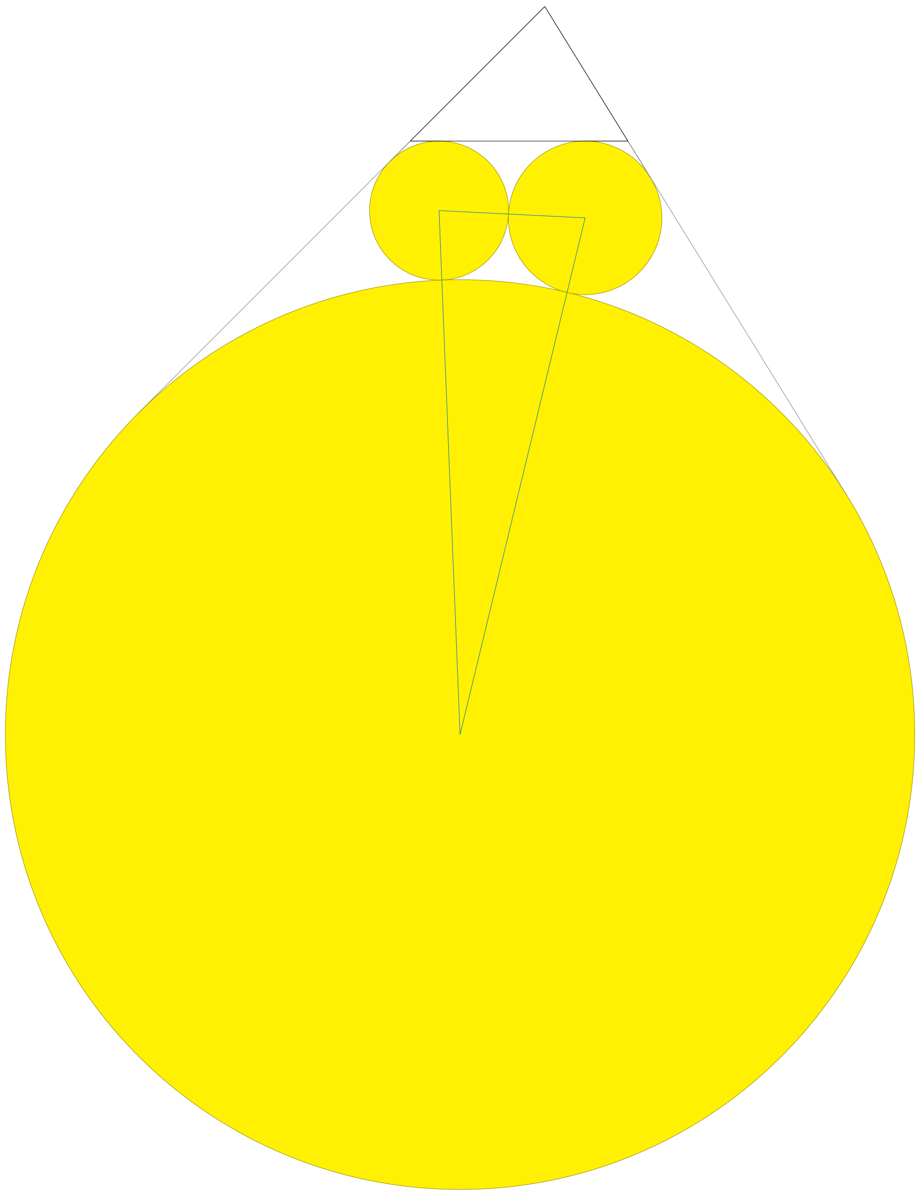


Figure 30: $\begin{cases} r_1 = r_B \sinh^2(\sigma_C - \beta_C), \\ r_2 = r_A \sinh^2(\sigma_C - \alpha_C), \\ r_3 = r \cosh^2 \sigma_C. \end{cases}$

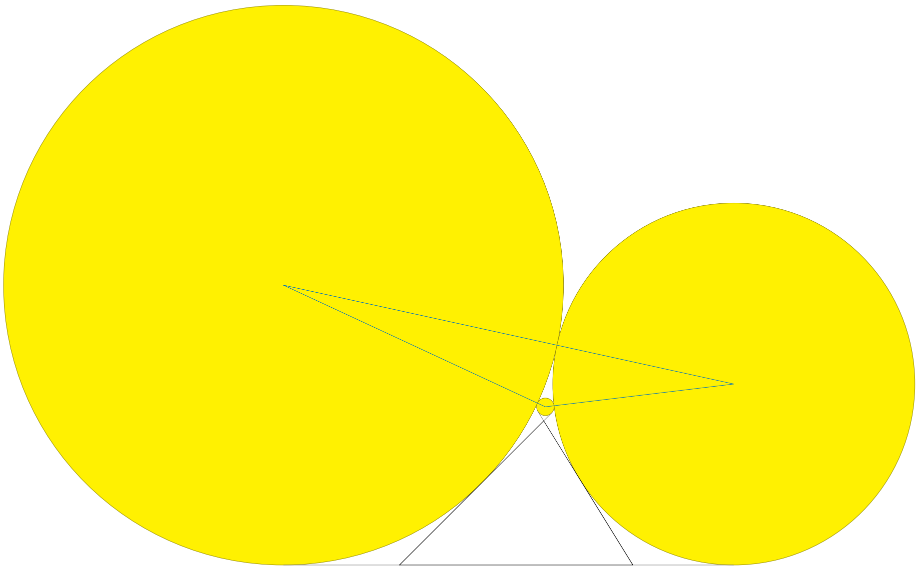


Figure 31: $$\begin{cases} r_1 = r_B \cosh^2(\sigma_C - \alpha_C), \\ r_2 = r_A \cosh^2(\sigma_C - \beta_C), \\ r_3 = r \sinh^2(\sigma_C - \gamma_C). \end{cases}$$

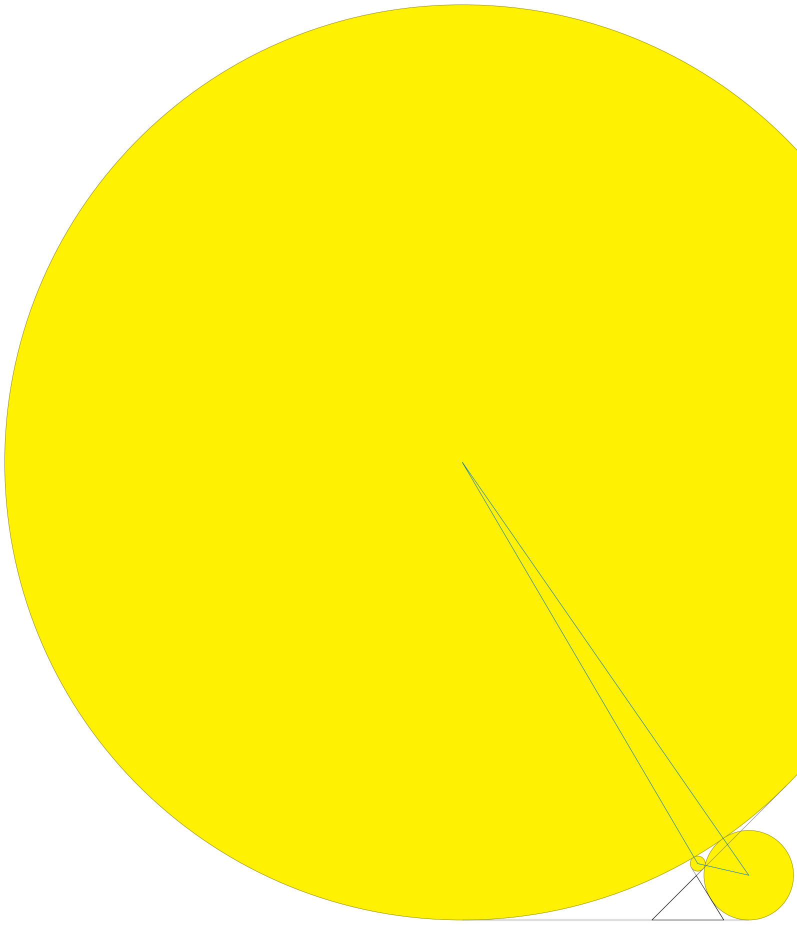


Figure 32: $\begin{cases} r_1 = r_B \cosh^2 \sigma_C, \\ r_2 = r_A \cosh^2(\sigma_C - \gamma_C), \\ r_3 = r \sinh^2(\sigma_C - \beta_C). \end{cases}$

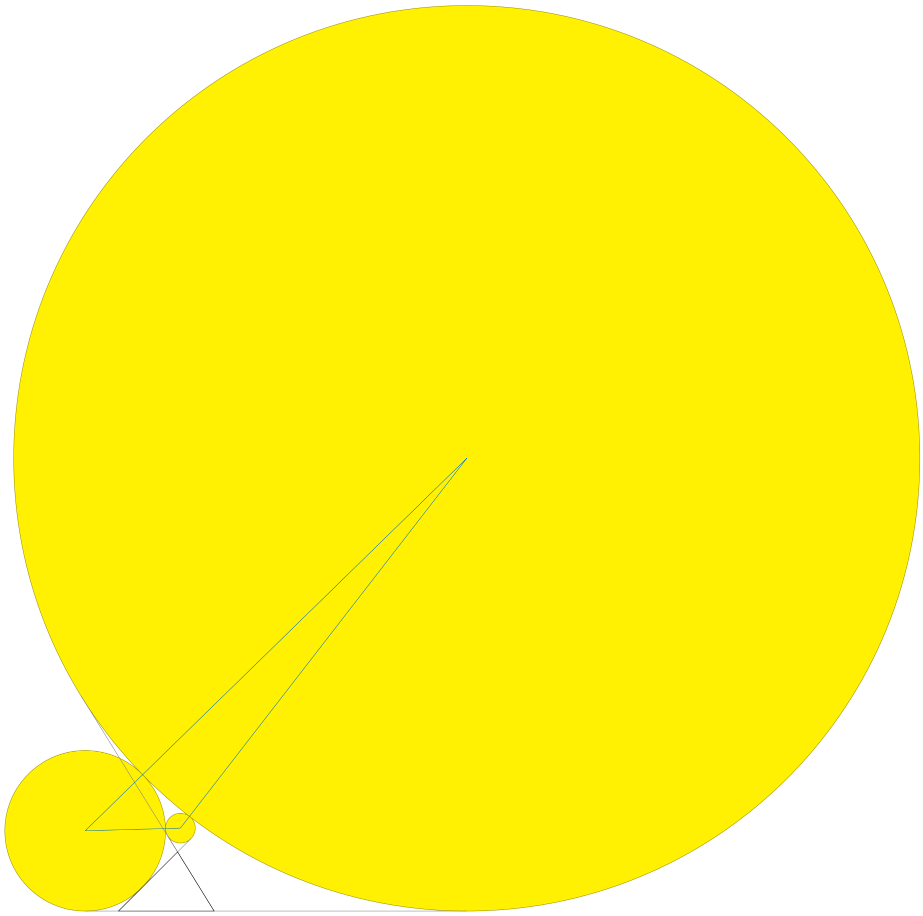


Figure 33: $\begin{cases} r_1 = r_B \cosh^2(\sigma_C - \gamma_C), \\ r_2 = r_A \cosh^2 \sigma_C, \\ r_3 = r \sinh^2(\sigma_C - \beta_C). \end{cases}$

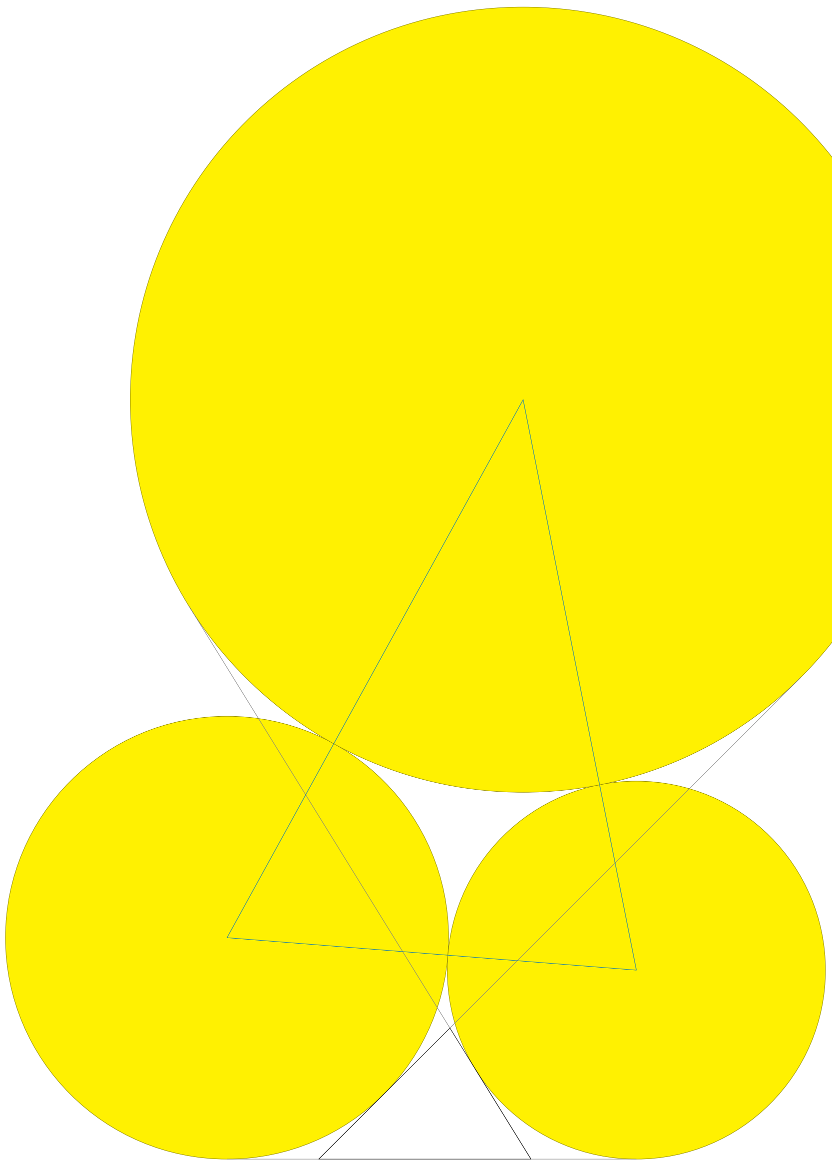


Figure 34: $\begin{cases} r_1 = r_B \cosh^2(\sigma_C - \beta_C), \\ r_2 = r_A \cosh^2(\sigma_C - \alpha_C), \\ r_3 = r \sinh^2 \sigma_C. \end{cases}$